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ABSTRACT

The purpose of this text is to teach learning and understanding of mathematics at grades seven through nine through the use of science experiments. Previous knowledge of science on the part of students or teachers is not necessary. The text is designed to be usable with any mathematics textbook in common use. The material can be covered in four weeks. Chapters in the text include: (1) Open Sentences and Equations; (2) An Experimental Approach to Linear Functions; and (3) Trampolines and Gases. The appendices contain sections on graphing, scientific notation, and the metric system. A glossary is also included. (RH)

MATHEMATICS THROUGH SCIENCE

Part II: Graphing, Equations and Linear Functions

Student Text

(revised edition)

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PREFACE

Most of the mathematical techniques that are in use today were developed to meet practical needs. The elementary arithmetic operations have obvious uses in everyday life, but the mathematical concepts which are introduced in the junior high school level and above are not as obviously useful.

The School Mathematics Study Group has been exploring the possibility of introducing some of the basic concepts of mathematics through the use of some simple science experiments. Several units were prepared during the summer of 1963 and were used on an experimental basis in a number of classrooms during the following year. On the basis of the results of these trials, these units were revised during the summer of 1964.

This text is designed to be usable with any mathematics textbook in common use. It is not meant to replace the textbook for the course, but to supplement it. Previous acquaintance with science on the part of the student is unnecessary. The scientific principles involved are fairly simple and are explained as much as is necessary in the text. Each experiment opens a door into a new domain in mathematics: linear functions, graphs, translation of axes, the distributive property, and the solution of equations. We hope that student learning and understanding will be improved through the use of this material.

The experiments have all been done in actual classroom situations. Every effort has been made to make the directions for the experiments as clear and simple as possible. The apparatus has been kept to a minimum.

The writers sincerely hope that this approach to mathematics will prove both useful and interesting to the student.

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Chapter 1

OPEN SENTENCES AND EQUATIONS

1.1 Introduction

In this chapter we shall perform an experiment to learn how to solve problems by experimentation. Everyone has played on a seesaw at one time or another. You know that a small boy sitting at one end of a seesaw can balance a big boy who sits closer to the fulcrum on the other side. The fulcrum is the point at which the seesaw is supported.

Can you tell exactly how much closer to the fulcrum the big boy must sit to balance the mass of the small boy at the other end of the seesaw? Do you think that the masses of the boys and their distances from the fulcrum are related somehow? To be able to answer these questions properly, we must know more about the seesaw.

In our first experiment, we will set up a miniature seesaw. In observing how the seesaw operates, we will discover a rule which will explain the way it works, and then try to state this rule in mathematical form.

1.2 The Seesaw Experiment

A simple model of a seesaw can be constructed from a meter stick, spring clamp, triangular file and Dixie cups. Make a support by placing two six-ounce Dixie cups upside down on a block which is about 8 inches long. In order to be sure that the meter stick is supported at its midpoint, clamp a spring paper clamp with the 50 cm mark as close to the center of the clamp as possible. Insert a 5-inch triangular file in the hole of the clip so that one edge of the file is in the upright position. (See Figure 1.)

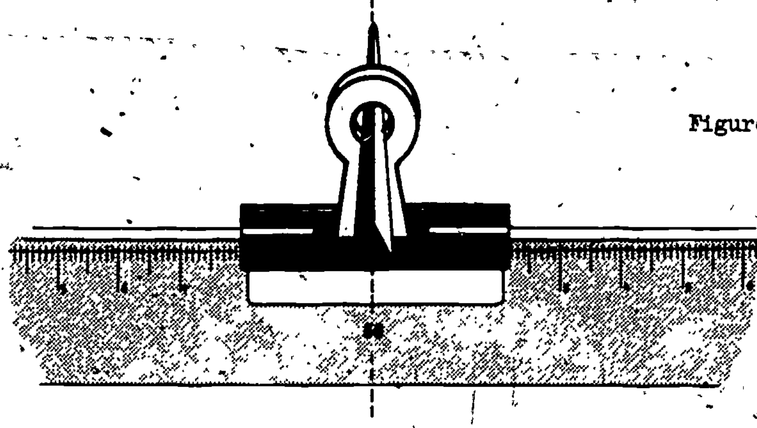


Figure 1

The cups serve as a support when the triangular file is placed across them. (See Figure 2.) The stick should settle in a horizontal position. If it does not, place small pieces of modeling clay on the end of the lighter side until it does balance. Let us agree that the meter stick is balanced when it comes to rest in a horizontal position. We need some device to indicate when the stick has returned to the horizontal position after having been moved.

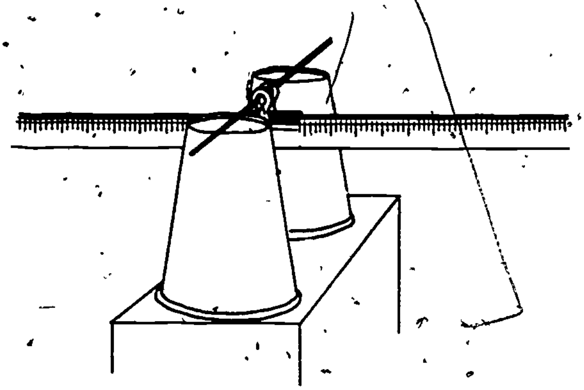


Figure 2

To determine when the stick is horizontal, we shall use two $\frac{3}{8}$ - inch dowels which are at least 8 inches long and two chunks of modeling clay. Stick each dowel into one of the chunks of clay so that the dowels stand in a vertical position. (See Figure 3.) Place one of the upright dowels behind the balanced meter stick near the right-hand end of the meter stick. Place

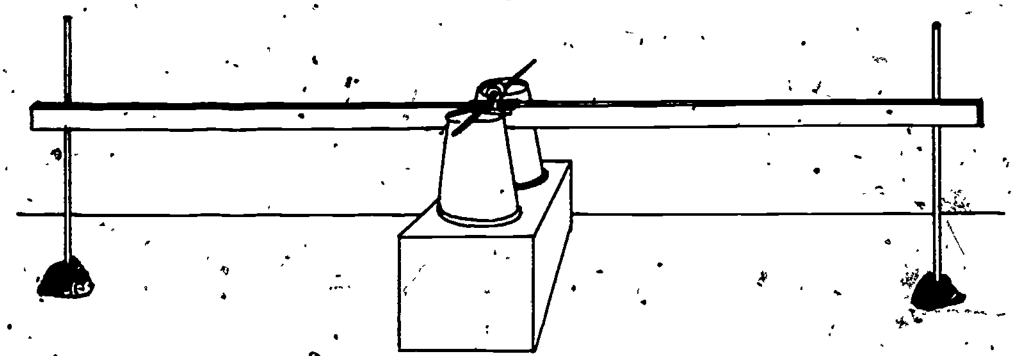


Figure 3

the other upright dowel in a similar position near the left-hand end of the meter stick. Now place a small pencil mark on each dowel stick so that the pencil mark and the top edge of the meter stick are in a horizontal line of sight. Measure the distance from the table top to the pencil mark on each dowel. When these distances are equal, the meter stick is horizontal. Bend paper clips to serve as hangers for the weights. Open the paper clips so they will slide easily on the meter stick. For this part of the experiment we shall need 10 grams (2 weights), 20 grams (2 weights), 50 grams, 100 grams, 200 grams (2 weights).

Begin the experiment by hanging a mass of 20 gm on each side of the fulcrum 30 cm from the fulcrum. (See Figure 4.) Does the stick balance? Slide each 20-gm mass to a position 10 cm from the fulcrum. Does the stick balance again? Now move each mass to 40 cm from the fulcrum. Do you notice that when we place objects with equal masses on opposite sides of the fulcrum at equal distances from it, the stick always balances?

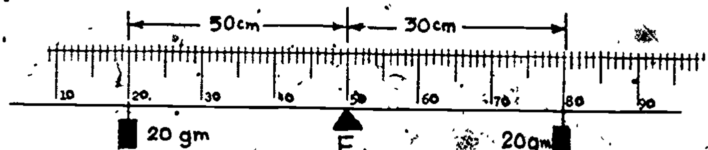


Figure 4

An experimenter may wonder, however, what might happen if the masses on either side of the fulcrum are not equal. To answer this question keep the 20-gm mass at 30 cm from the fulcrum on the left side. Attach a 30-gm mass on the right side, 30 cm from the fulcrum. (See Figure 5.) There is no balance now, since the right side of the stick tips down. Can you explain why?

Now slide the 30-gm mass closer to the fulcrum until the stick is balanced. At what distance does the mass of 30 gm balance the mass of 20 gm placed 30 cm from the fulcrum? Do you find that the smaller mass placed farther away from the fulcrum balances the larger mass placed closer to the fulcrum?

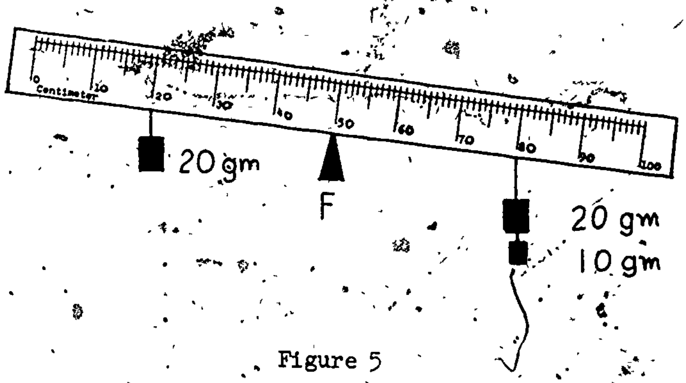


Figure 5

Choose two objects with different masses and place one mass on each side of the fulcrum. Now slide them back and forth until you get a balance. For instance, use 20 gm on one side and 50 gm on the other. Did you notice that no matter what pair of masses you use you can always balance the stick by placing the masses at the right distances from the fulcrum?

These observations lead to the conclusion that hanging objects with equal masses on each side of the stick at equal distances from the fulcrum will make the stick balance. Different distances are needed to achieve a balance when the masses are unequal.

Our purpose is to find a general rule which describes the relationship between the mass and the distance from the fulcrum so we can tell in advance where to place one object of known mass to balance another object of known mass.

To establish this relationship, further experimentation is needed. Therefore, perform several trials (experiments) in a variety of situations.

To keep the experiment simple, use the same object (200 gm) in all the trials at a fixed distance (6 cm) to the right of the fulcrum. (See Figure 6.) Then balance it in turn on the left-hand side with 120 gm, 60 gm, 30 gm, 40 gm and 200 gm. Slide each object on the left back and forth until the stick is balanced.

For convenience, let "m" represent the mass of any object that is hung on the stick and "d" the measure of the distance from the fulcrum.

Perform the first trial. Remember we place 200 gm at 6 cm to the right of the fulcrum. (See Figure 6.) Hang the 120-gm mass on the left-hand side. Slide it back and forth until you find the distance from the fulcrum at which the stick balances. Then read this distance to the nearest cm and record it in the first row of your table. (See Table 1.)

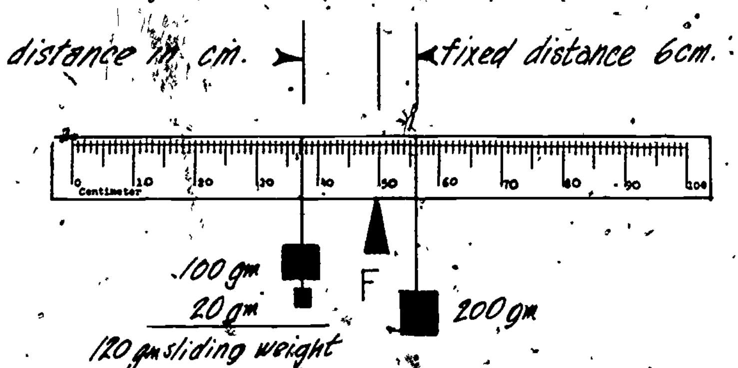


Figure 6

Balanced Meter Stick

	Left side		Right side	
Trials	Mass in grams m	Distance from fulcrum in cm d	Mass in grams m	Distance from fulcrum in cm d
I	120		200	6
II	60		200	6
III	30		200	6
IV	40		200	6
V	80		200	6

Table 1

In the second trial, keep the 200-gm mass at 6 cm from the fulcrum as was done in the first trial, and hang the 60-gm mass on the left side. 60 grams is half the mass of 120 gm. How far away from the fulcrum must the 60-gm mass be to get a balance? Check your guess by reading off the distance. Enter this distance in the second row.

Repeat the same procedure with 30 gm, 40 gm and finally with 80 gm. Be sure the stick is exactly in a horizontal position before you read the distance on the left-hand side. As you perform the last three trials, do not forget to read the corresponding distances and record them in your table.

Study the numbers recorded in the left side of your table. Is there any connection between the mass of the objects and their corresponding distances from the fulcrum? Compare the mass and distance in the first row with the mass and distance in the second row. Notice that the value of m decreased to one half its original mass, namely from 120 gm to 60 gm, and at the same time the value of the corresponding distances doubled from 10 cm to 20 cm. Compare the numbers in the second and third trials. The distance d changes as the mass was decreased from 60 gm to 30 gm. What is the ratio between the masses; what is the ratio between the matching distances?

The table shows five different pairs of masses and distances which make the stick balance. For instance, in the first trial the mass of 120 gm at 10 cm has the same effect as the mass of 200 gm at 6 cm. How are the numbers in these pairs related? You might guess that the product of 120 × 10 equals the product of 200 × 6, or, $120 \times 10 = 200 \times 6$.

In the second trial the mass of 60 gm at 20 cm has the same effect as the mass of 200 gm at 6 cm. The product 60 gm × 20 cm = 200 gm × 6 cm again.

Do you find that the product of the mass and its distance on one side must be equal to the product of mass and distance on the other whenever the stick balances?

Let's check:

	Left side		Right side	
Trial	Mass	Distance	Mass	Distance
III	30 × 40 = 1200		200 × 6 = 1200	
IV	40 × 30 = 1200		200 × 6 = 1200	
V	80 × 15 = 1200		200 × 6 = 1200	

After proceeding down the table and checking all entries, we conclude the following: "Whenever the stick balances, the product of mass and distance on one side equals the product of mass and distance on the other side."

In this experiment, whenever the stick balanced, the product of the masses and corresponding distances was always 1200. The table verifies this result. This result can be put in mathematical form by the sentence $m \times d = 1200$, where "m" represents the mass of the object on the left-hand end of the meter stick, and "d" represents its distance from the fulcrum.

In the experiment, the product of the measures of the mass and its distance was 1200 in all cases because we were always balancing objects on the left side with the mass of 200 gm at 6 cm to the right of the fulcrum. This meant that when an object was suspended on the left side, it was necessary to slide this object along the meter stick until the mass of the object times its distance from the fulcrum became equal to 1200.

Our experiment represents only <u>one</u> example of how to balance a seesaw. Any two objects can be balanced on a seesaw, provided that the product of the mass and the distance on one side is equal to the product of the mass and the distance on the other side. To verify this statement, consider the following problems:

(1) At 10 cm from the fulcrum, how large a mass will balance a mass of 30 gm which is 20 cm from the fulcrum?
Following our rule: $30 \times 20 = ? \times 10$. This is satisfied by a mass of 60 gm. Check it on your meter stick instrument.

(2) Where should we place 300 gm to balance 90 gm placed 20 cm from the fulcrum? Find the distance by suspending 300 gm from the stick on one side and 90 gm at 20 cm on the other until you get a balance. The distance is obviously 6 cm, since $90 \times 20 = 300 \times \underline{6}$.

In summary, we conclude from our experiments that if the product of the mass and distance on one side of the fulcrum is equal to the product of the mass and distance on the other side, then the seesaw balances.

Exercise 1

1. Below is a table of values from an experiment with a seesaw. Masses were hung on the right-hand side to balance the 6 pounds at 8 cm to the left of the fulcrum. Find where we should place the masses shown in the table to balance 6 pounds placed at 8 cm from the fulcrum.

Left side			Right side	
m	d		Mass of objects in pounds m	Distance from the fulcrum in cm d
6	8		12	4
6	8		2	
6	8		8	
6	8		24	
6	8		16	
6	8		6	

2. Find the values for the masses and distances in the given table if you want to balance 20 gm at 14 cm from the fulcrum on the other side.

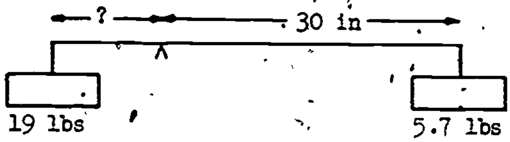
m gm	20	40	10			50
d cm				15	30	

3. How far from the fulcrum should a 20-gm mass be placed on the left side to balance a 40-gm mass placed 20 cm from the fulcrum on the right side?

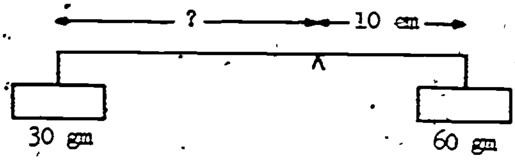
4. A boy, whose mass is 70 lbs, rode a seesaw with his father, whose mass is 175 lbs. If the father sat 4 ft from the fulcrum, where must the boy sit to balance the seesaw?

Find the missing values in Problems 5, 6 and 7.

5.



6.



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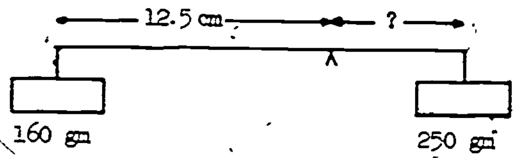


8. Is there a place on the seesaw where a single mass can be placed and a balance obtained? If so, what is the distance of the mass from the fulcrum?

1.3 Number Sentences

In the previous section we found how one can balance a seesaw supported at its center. Different masses were balanced with a fixed mass of 200 gm at 6 cm from the fulcrum. The results were recorded in tabular form. Then, from this table, a rule was developed and expressed in the mathematical form

$$m \times d = 1200$$

where "m" represents the mass of any object that is placed on the stick and "d" represents its distance from the fulcrum.

The mathematical language which uses the number sentence to state relationships is the language used by the scientist, engineer, mathematician and others to communicate ideas to one another. The number sentence

$$m \times d = 1200$$

is an illustration of this form. This form allows great quantities of information to be stated in a simple manner. For example, $m \times d = 1200$ is a representation of all the data found in the previous experiment and in Exercise 1. Furthermore, it was not necessary to set up an experiment for each problem in Exercise 1 after the relationship was determined. Note, too, that in writing this sentence only five symbols are used, "m", "x", "d", "=" and "1200".

Just as sentences are used in talking and writing to discuss our everyday experiences, so sentences are used in science and mathematics to describe and explain. For instance, you are familiar with the following sentence: The diameter (D) of a circle equals twice the radius (r). This can be stated in mathematical form as "$D = 2r$". This is just as good as "Johnny is sleepy today". The former sentence states the fact that in any circle, <u>the diameter is twice the radius</u>. The latter sentence states the fact that Johnny is sleepy today.

There are many other sentences that make statements about numbers and quantities. However, not all sentences make statements about quantities that are equal. For example, "Five is greater than three". As a number sentence this is written $5 > 3$. The symbol $>$ is read "is greater than". Likewise, the symbol $<$ is read "is less than" and is used in number sentences such as "Three is less than five", written $3 < 5$. Another symbol sometimes used is read "not equal to", and written $\neq$. The set of symbols, $=$, $\neq$, $>$, $<$ are the verb phrases commonly used in writing mathematical sentences. These verb phrases state the relationship involved between the word phrases. You are familiar with sentences such as "The sum of a number, x, and eight is twelve". This verbal sentence can be stated in mathematical form by saying $x + 8 = 12$. Similarly,

Nine is greater than the sum of three and four:	$9 > 3 + 4$
The product of three and five is fifteen:	$3 \cdot 5 = 15$
Twenty-one is less than the sum of eight and fifteen:	$21 < 8 + 15$
The product of a certain number y and three is not equal to six:	$y \times 3 \neq 6$

These sentences are examples of <u>number sentences</u>.

For example, the first sentence, $x + 8 = 12$, consists of two expressions, "$x + 8$" and "12". These expressions are not sentences, they are only parts of a sentence and are called <u>phrases</u>. Then the difference between a sentence and a phrase is that <u>a phrase does not make a statement</u> but <u>a sentence does</u>.

1.4 <u>Number Phrases</u>

Let us return to the first two examples of phrases, $x + 8$ and 12. Notice that 12 represents only one specific number whereas $x + 8$ can represent any number, depending on the value of x. For instance,

if x is 3, then x + 8 represents 11

if x is 8, then x + 8 represents 16.

A number phrase is a name for a number. If the phrase represents a specific number, it is called a closed number phrase, or more simply, a closed phrase. For example, 19, (3 + 2), $\frac{18}{3}$, 2(3 + 0.5), (4 × 7 - 1), etc. are closed phrases.

Number phrases which do not represent a specific number are called open number phrases, or more simply, open phrases. The value of the phrase depends on what number the symbol in the open phrase represents. For example, 3x + 2 represents 5 if x is 1, but it represents 14 if x is 4 and 32 if x is 10.

Exercise 2

1. Translate each of the following number phrases into mathematical symbols:
   (a) The sum of the number x and 15.
   (b) The product of 8 and x.
   (c) One fourth of the number x.
   (d) A number which is 4 less than x.
   (e) The division of 18 by x.
   (f) Three greater than x.
   (g) One less than two thirds of x.
   (h) The number x less than 23.
2. For each of the number phrases in Problem 1 find the number represented by the phrase if x = 12.

1.5 Parentheses

Assume you are faced with a problem such as the following:

"Find the number represented by the open phrase 6 + 8n, if n is 4."

Then, replacing 4 for n we get 6 + 8 × 4. This is a numerical phrase. What number does it represent? If you look at it one way, you might say,

6 + 8 is 14 and 14 × 4 is 56.

Therefore, the numerical phrase 6 + 8 × 4 could represent 56. However, if we look at the phrase another way, reading it from right to left,

8 × 4 = 32 and 6 + 32 = 38.

Therefore, there seem to be two possible answers. In order to eliminate the possibility of calculating the value of an open phrase in various ways, the following mathematical rule is defined:

"In any given expression where there is a multiple operation, we agree to multiply and divide, before we add or subtract."

Applying this rule to the above example, we find the value of $6 + 8 \times 4$, by first multiplying 8 by 4 which is 32 and then adding 6. Therefore, the correct answer is 38.

Illustrative examples:

1. How would you find the value of $8 - 12 \div 2$? Here the division is done before the subtraction, so $12 \div 2 = 6$ and $8 - 6$ is 2.

2. Find the value of the closed phrase $4 \times 3 - 6 \div 2 + 1$. Remember in this case the multiplication and division are done first. Therefore, $4 \times 3 = 12$ and $6 \div 2 = 3$. So $4 \times 3 - 6 \div 2 + 1$ can be simplified to $12 - 3 + 1$ which is 10.

3. Consider a problem from arithmetic. Subtract 2 from 15 and add 3 to the difference. Translated into mathematical form, $15 - 2 + 3$. This problem involves only subtraction and addition (no multiplication or division in it). In simplifying, you can take your choice. You can read from left to right, $15 - 2$ is 13 and $13 + 3$ is 16, or you can start to read from right to left, $-2 + 3$ is $+1$ and $15 + 1$ is 16. Both ways give the same number, 16.

In working with the phrase $15 - 2 + 3$, we must be careful to subtract only the 2 from 15, and not the sum of 2 and 3. To avoid similar confusion, we use symbols "( )", called <u>parentheses</u>. This means that when we enclose a numerical phrase such as $5 + 4$ in <u>parentheses</u>, we intend that the phrase "$(5 + 4)$" be treated as a single number. For example, to subtract the sum of 4 and 3 from 18, you write $18 - (4 + 3)$. That is, $(4 + 3)$ is treated as a single number, 7, and subtracting the 7 from 18, we get 11.

Using parentheses, there should be no difficulty translating the following problem: "Multiply the sum of 2 and 6 by 4" into the symbolic form $(2 + 6) \times 4$. Treating (2×6) as a single number, you get 8, and 8×4 is 32.

On the other hand, working with a phrase like $2 + 6 \times 4$, parentheses are not needed since we agree that multiplication comes first before addition. Therefore, the product of 6 and 4 is 24, and $2 + 24 = 26$. Notice that the two phrases, $(2 + 6) \times 4$ and $2 + 6 \times 4$, represent two different numbers. The first is 32, and the second is 26. One final note: a numerical phrase like $(2 + 6) \times 4$ is often written without the symbol "$\times$", as in $(2 + 6)4$. Here the operation of multiplication is implied.

These expressions are but two methods used for indicating the operation of multiplication. We have already seen this done in such expressions as 2r, which is read, "two times the number represented by r". Similarly, when we write (2 + 6)4, it is understood that 4 is multiplied by (2 + 6).

The most common forms for indicating multiplication are as follows:

(a) $2 \times r$
(b) $2r$
(c) $2(r)$
(d) $2 \cdot r$
(e) $(2)(r)$

Form (b) is not acceptable when r is a numeral. For example, 28 means twenty-eight, not 2 times 8. The dot of form (d) would be used in this case as $2 \cdot 8$. In Section 1.3 the number sentence $m \times d = 1200$ could have been written $md = 1200$ if form (b) is used.

There are fewer expressions used for division, only 3 forms being commonly used. Everyone is familiar with the form $r \div 2$, which is read, "the number represented by r divided by 2". The other two forms use short line segments. One is $\frac{r}{2}$ and the other $r/2$.

Exercise 3

1. Which of the following closed phrases name the same number?

(a) $2 + 4 \times 5$ and 22
(b) $(2 + 4)5$ and 30
(c) $2 + (4 \times 5)$ and 30
(d) $4 + 3 \times 2$ and $(4 + 3)2$
(e) $5 \times 8 + 3$ and $(5 \cdot 8) + 3$
(f) $32 \div 8 - 4$ and $6 \times 4 + 5$

2. Place parentheses in the following so that

(a) $2 \times 3 + 1$ represents 8
(b) $2 + 4 \times 3$ represents 14
(c) $6 \times 3 - 1$ represents 17
(d) $12 - 1 \times 2$ represents 22
(e) $18 - 6 + 3$ represents 16

3. Find a number for each numerical phrase:

(a) $5 \times 8 + 7$

(b) $5(8 + 7)$

(c) $(9 + 1)(3 + 4)$

(d) $6 + 2 \cdot 4$

(e) $14 - 3 \times 2$

(f) $(17 - 6)4$

(g) $\left(\frac{6 + 2}{4}\right) + 5$

(h) $9(1 + 3) - (8 + 2)$

(i) $9(1 + 3) - 8 + 2$

4. Using parentheses, rewrite the following closed phrases so they represent the same number. For instance, $2 \times 5 + 6 \times 2$ can be written $(2 \times 5) + (6 \times 2)$ and both represent 22.

(a) $3 + 8 - 4$

(b) $\frac{1}{2} \times 6 + 4$

(c) $3 \times 5 - 4 \times 2$

(d) $36 \div 9 + 5 - 2$

1.6 Distributive Property of Numbers

Another property of numbers can be described by using parentheses. How would you solve the following problem in the simplest possible way?

> A meat market sells steak for \$1.20 a pound. A woman bought two steaks; one weighed 3 lbs, and the other, 2 lbs. The total cost of the two steaks can be computed in two ways:
>
> (1) $3 \times \$1.20 = \$3.60$, the cost of the larger steak
>
> $2 \times \$1.20 = \underline{\$2.40}$, the cost of the smaller steak
>
> $\$6.00$, total cost

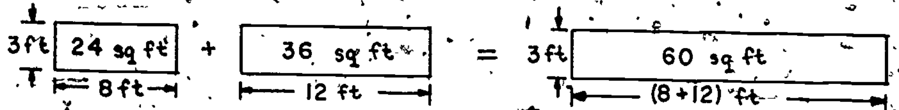
or, (2) Take the total weight, which is $(3 + 2)$ pounds, and multiply it by \$1.20.

$(3 + 2)(\$1.20) = 5 \times \$1.20 = \$6.00$

Do you agree that in computing the cost, the second method is simpler?

Consider another problem:

> Two strips of a carpet, one measuring 3 ft × 8 ft, the other 3 ft × 12 ft, are sewed together to make one runner, or a single piece. How many square feet are there in the runner?



From the illustration, it can be seen that the sum of the two pieces is equal to the final pieces,

$$3 \times 8 + 3 \times 12 = 3 \times (8 + 12)$$

that is, the area of the first piece plus the area of the second piece is equal to the area of the runner.

These problems illustrate the <u>distributive property of multiplication over addition</u>. This property can be further illustrated by the following product:

7×13	This may be written as
$7(10 + 3)$.	Notice that 10 + 3 in the parentheses requires multiplication of the sum. This product is
$= 7 \times 10 + 7 \times 3$	This indicates that the multiplication has been distributed over each term in the addition.
$= 7 + 21$	
$= 91$	

The name "distributive property of multiplication over addition" is usually shortened to "distributive property". We can state it in a general form as follows: "For every number a and every number b and every number c,

$$a(b + c) = a \times b + a \times c".$$

Study the following examples carefully. In one of them, the distributive property applies; in the other, it does not.

$8(3 + 2) = (8 \times 3) + (8 \times 2)$ Here the distributive property <u>does</u> apply. Multiplication is distributed over addition.

$8 + (3 \times 2) \neq (8 + 3) \times (8 + 2)$ Here the distributive property <u>does not</u> apply. Addition is not distributed over multiplication.

The distributive property of multiplication over addition is frequently used in mathematics.

In the following examples, to further your understanding of distributive property, compare the value of the indicated product with the value of the indicated sum.

Indicated Product		Indicated Sum
$8(4 + 3)$	=	$8(4) + 8(3)$
$3(100 + 20)$	=	$3(100) + 3(20)$
$5(26)$	=	$5(20) + 5(6)$
$6 \times 5\frac{1}{3}$	=	$6(5) + 6(\frac{1}{3})$

This illustrates the changing of an indicated product to an indicated sum. Here is another example:

$$4(10 + 2) = 4(10) + 4(2).$$

It is also correct to change an indicated sum to an indicated product by use of the distributive property. For example,

$$4(10) + 4(2) = 4(10 + 2).$$

Again, compare the value of the indicated sum with that of the indicated product in the following examples:

Indicated Sum		Indicated Product
$15(8) + 15(2)$	=	$15(8 + 2)$
$21(7) + 21(3)$	=	$21(7 + 3)$

Exercise 4

1. Which of the following problems are indicated sums and which are indicated products?

   (a) $3(8 + 5)$
   (b) $3(8) + 3(5)$
   (c) $2(6) + 2(3)$
   (d) $4(3 + 6)$
   (e) $7 + (3 \times 6)$
   (f) $(7 + 3)6$

2. Express the following indicated products as indicated sums and indicated sums as indicated products:

   (a) $4(47 + 3)$
   (b) $9(34 + 6)$
   (c) $\frac{2}{3}(8) + \frac{2}{3}(4)$
   (d) $18(3.2) + 18(.8)$

3. Perform the indicated operations the easier way. Show your method.

Illustrative Example:

$$110(8) + 110(92) = 110(8 + 92) \text{ or } 110(8) + 110(92) = 880 + 10120$$
$$= 110(100) \qquad\qquad = 11000$$
$$= 11000$$

(a) $12(\frac{1}{3} + \frac{1}{4})$

(b) $\frac{1}{5}(\frac{7}{8}) + \frac{1}{5}(\frac{1}{8})$

(c) $9(11 + 9)$

(d) $0(17 + 83)$

(e) $\frac{8}{9}(0 + 9)$

4. Show how you could use the distributive property to perform the multiplication mentally.

Example:

$$6 \times 24 = 6(20 + 4)$$
$$= 6(20) + 6(4)$$
$$= 120 + 24$$
$$= 144$$

(a) $7(22)$

(b) $12(33)$

(c) $15(36)$

1.7 Translation of Open Phrases to Word Phrases

Numbers are often used in talking about things. For instance, the number three can refer to 3 books, 3 inches, 3 apples, etc. This does not mean that "3 books" is a number. In the same way, any number, n, can be used to talk about things like n books, n inches, etc. Remember when we say "n books", we mean that n is the number of books. Similarly, the translation of an open phrase like $2x + 3$ to a word phrase depends on what meaning we give to x. The number 3 must be given the same meaning as that given to x. For instance, the phrase $2x + 3$ can be translated in the following ways:

	x	$2x$	$2x + 3$
(a)	number of points Mary made in a game	number of points Sue made if she made twice as many as Mary	number of points Sue made if she made 3 more than twice as many as Mary
(b)	number of books Jim has	number of books Peter has if he has twice as many as Jim does	number of books Peter has if he has 3 more than twice as many as Jim does

As another example, the phrase $\frac{1}{2}a - 4$ can be translated as follows:

	a	$\frac{1}{2}a$	$\frac{1}{2}a - 4$
(c)	length of a rectangle	the length of a rectangle if its length is half that of the original rectangle	the length of a new rectangle if its length is 4 units less than half that of the original rectangle
(d)	distance from city A to city B	distance from city A to city C if its distance is half that of the distance from city A to city B	distance from city A to city D if its distance is 4 miles less than half the distance from city A to city B

These are the two translations of each of two phrases. Many more translations are possible for each phrase.

Exercise 5

1. Can you think of a different way to translate the phrase $2x + 3$ into a word phrase?
2. How many translations of $3x - 5$ can be made? Give examples.

3. In the following problems, write a translation of the phrase to a verbal phrase:

(a) $n + 6$
(b) $n - 6$
(c) $2n$
(d) $2n + 1$
(e) $\frac{n}{3}$
(f) $\frac{n + 1}{3}$
(g) $x + 7x$
(h) $2x + 3x$

1.8 Translation of Word Phrases to Open Phrases

In the last section open phrases were translated into word phrases. We noticed that there was not a single translation but many possible translations. For instance, the translation of the phrase $2n + 3$ depended on the meaning assigned to the symbol n.

It is also possible to go the other way, and translate word phrases into open phrases.

Suppose you want to talk about your age 5 years from now. This is easy since you know your age. You might reason as follows:

> The number of years in my age now is 13; then 5 years from now my age will be $\underline{13 + 5}$ years. So, I can say that in 5 years my age will be 18 years.

Let us say you want to talk about Bill's age 5 years from now. Suppose you do not know his age for sure. Then you would say that the number of years in Bill's age now is $\underline{x}$; consequently, the number of years in Bill's age 5 years from now is $\underline{x + 5}$ years. Notice the phrase "$x + 5$" represents the number of years in Bill's age 5 years from now. In this problem a word phrase has been translated (the number of years in Bill's age 5 years from now) into the symbolic phrase $(x + 5)$.

How would you translate an expression such as "four more than a number y" into a symbolic phrase? In thinking about this expression, you could say that we begin with a number y and add 4 to it. This suggests that we write $y + 4$.

Consider the following word phrase: "A line segment 3 feet longer than another line segment". Our purpose is to write this word phrase as an open phrase. The number of feet in the first line segment is unknown. Let "f" represent the number of feet in the first segment. Then "$f + 3$" represents the number of feet in the second segment.

Exercise 6

1. Translate the following word phrases to symbols:
   - (a) If the number of years in Bill's age is now K, what is the number of years in Bill's age 7 years from now?
   - (b) The number of cents in x quarters
   - (c) The number of cents in x dollars
   - (d) The number of years in Sam's age 3 years ago
   - (e) The number of years in John's age 4 years from now
   - (f) The number of feet in y yards
   - (g) The number of inches in b yards

2. Translate each of the following word phrases to symbolic phrases:
   - (a) The sum of a number x and 2
   - (b) The number x decreased by 8
   - (c) The number x subtracted from 15
   - (d) The product of 7 and x
   - (e) The quotient of a number 3 divided by x
   - (f) The number x increased by 6
   - (g) The number x divided by 2
   - (h) One third of a number x.

3. For each of the number phrases in Problem 1, find the number represented by the phrase if the unknown number is 24.

4. Write open phrases to represent each of the following:
   - (a) The sum of an even number and the next even number
   - (b) One half of the sum of a number and 6
   - (c) Seven less than 3 times a number
   - (d) Twice a number increased by 3
   - (e) Twice the sum of 7 and 2
   - (f) Find the total age of Mary and Sue if Mary is 5 times as old as Sue is.

     (Hint: Let x represent the number of years in Sue's age.)
   - (g) The number of cents Mike has, if he has x nickels and twice as many dimes as nickels

5. If the sum of the numbers t and 3 is doubled, which of the following phrases would be a correct name for the sum?

$$2t + 3 \text{ or } 2(t + 3)$$

6. If 5 is added to twice a certain number n and the sum is divided by 3, which phrase is the correct name for the quotient?

$$\frac{2n+5}{3} \quad \text{or} \quad \frac{2n}{3}+5$$

7. If one fourth of a certain number x is added to one third of four times the same number, which phrase is the correct name?

$$\frac{1}{3}(4x) + \frac{1}{4}(x) \quad \text{or} \quad \frac{4}{3}(x) + \frac{1}{4}(x)$$

8. If the number of gallons of milk purchased is y, which is the correct phrase for the number of quart bottles that will contain it?

$$4y \quad \text{or} \quad \frac{y}{4}$$

9. If a is the number of feet in the length of a certain rectangle and b is the number of feet in the width of the same rectangle, which phrase is the correct name for the perimeter?

$$2(a - b) \quad \text{or} \quad ab$$

Fill in the blanks in the following problems:

10. If k represents a number of kilometers, then the phrase __________ represents the number of meters in k kilometers.

11. A mathematical phrase indicating the number of centimeters in s meters is __________.

12. Given a symbol d representing the number of liters in a container, the phrase __________ represents the number of milliliters in that container.

13. The number of grams in p milligrams is __________.

14. The number of grams in t kilograms is __________.

15. Therefore, the sum of t kilograms and w grams would be __________ grams.

16. The number of centimeters in k meters and n centimeters would be __________.

17. Adding t centigrams to s grams would result in a sum of __________ grams.

18. In a mixture made up of oxygen and nitrogen, there are 4 times as many oxygen molecules as nitrogen molecules. Write a mathematical phrase for the number of oxygen molecules if there are h molecules of nitrogen. __________

1.9 Numerical Sentences

In mathematics we use sentences to make statements about numbers. For instance, consider the following examples:

$$8 \times 4 = 30 - 2$$

$$2(15) \neq 31$$

$$5 > 3$$

All of these sentences involve only numbers. Sentences which make statements about numbers are called <u>numerical sentences</u>.

For example, the first sentence "$8 \times 4 = 30 - 2$" states that the number represented by "8×4" is the same as the number represented by "$30 - 2$". It is read "8 × 4 is equal to 30 − 2", and it is a <u>true sentence</u>.

On the other hand, "$3 + 6 = 14$" is also a sentence. This sentence makes the statement that the number $(3 + 6)$ is 14. It is apparent that $3 + 6 = 9$ and certainly is not 14. However, "$3 + 6 = 14$" is still a perfectly good sentence, but it is a <u>false sentence</u>.

Exercise 7

Indicate whether each of the following sentences is true or false:

1. (a) $3(15) = 3(10) - 3(5)$
   (b) $4(8) + 4(2) = 4(10)$
   (c) $4(2 - 3) = 4(2) - 5$
   (d) $13 - (4 \times 5) = (13 - 4)5$
   (e) $4 \times 6 - 3 = 5 \times 7 - 8$
   (f) $3 + 4 \times 5 - 9 = 2 \times 6 - 7$
   (g) $14 - 8 - 6 \times 8 = 18 \times 7 - 23$
   (h) $31 \times 23 - 42 \times 7 = (27 \times 8) - 7(16)$

1.10 Open Sentences

Consider the following:

(a) ____________ is a student in our class.
(b) ____________ is a season of the year.
(c) The color of her hair is ____________.
(d) The sum of ____________ and 7 is 18.

These incomplete sentences are examples of open sentences. They are neither true nor false until the sentence is completed. The set of elements that may be used to complete the sentence is called the domain of the open sentence. For example, suppose that the domain of (a) is the set of names of students of our class. Then, when the blank in (a) is replaced by any member of the domain, we get a true sentence. Suppose, however, that the domain in (b) is the set {spring, summer, September, winter}. If the blank in (b) is replaced by "September", we get a false sentence.

The set of elements in the domain of the open sentence which, after replacement, produces true sentences is called the truth set of the open sentence. Notice that the truth set depends on the domain. For example, if the domain for (d) is the set of numbers less than 10, then the truth set of (d) is empty. However, if the domain for (d) is the set of numbers greater than 10, the truth set of (d) contains only the number 11.

Those open sentences such as (d) which contain numbers or quantities will be of special interest. It is convenient to use a symbol, such as x or y or m, instead of the blank so that we may write (d) as follows: "The sum of x and 7 is 18." This is again an open sentence. ~~It is neither true nor~~ false until x is replaced by a member of the domain of the open sentence. A symbol, such as x or y or m or a blank, which can be replaced by any member of a given set is called a variable. The given set is called the domain of the variable.

Consider the following open sentence: "The sum of y and 6 is 14." It is not meaningful to discuss the open sentence until we specify the domain of y. Let the domain of y be all positive numbers. Then write the open sentence in symbolic form as

$$y + 6 = 14.$$

What is the truth set of this open sentence? For instance, let us guess that 3 is in the truth set. If we replace y by 3, do we get a truth statement? Obviously not, because $3 - 6 = 14$ is a false statement. A few trials will convince you that the truth set contains only the number 8.

The truth set of an open sentence is also called the solution set. Thus, the solution set of the open sentence $y + 6 = 14$ is the set whose only member is the number 8. We shall also say that 8 is the solution of the equation $y + 6 = 14$.

Suppose we want to solve (to find the truth set or solution set) of the open sentence $x + 1 > 5\frac{1}{2}$. If we assume x is a variable whose domain is the set of all real numbers, the open sentence would state that a number x increased by one is greater than $5\frac{1}{2}$. For what numbers does the open sentence become a true sentence? Test to see if $4\frac{1}{2}$ is in the solution set. A little thought tells us that it is not, because the statement $4\frac{1}{2} + 1 > 5\frac{1}{2}$ is not true. Certainly, any number less than $4\frac{1}{2}$ will also not be in the solution set. However, any number greater than $4\frac{1}{2}$ will be in the solution set because if it is increased by one, the sum will be greater than $5\frac{1}{2}$. For example, 4.6 is a solution because

$$4.6 + 1 > 5\frac{1}{2}$$

is a true sentence. Therefore, the set of numbers greater than $4\frac{1}{2}$ is the solution set of the inequality $x + 1 > 5\frac{1}{2}$.

Open sentences are not completely specified until the domain of the variable is given. Since in most mathematical questions the domain is the set of all real numbers, we shall frequently omit making any specific statement about the domain. We make the following agreement: If the domain is not specified, it is understood to be the set of all real numbers.

In physical problems, the domain cannot be the set of all real numbers. For example, in the seesaw problem we found $md = 1200$. Here, the domain of d is the set of positive numbers between 0 and 50, because the distance from the fulcrum can be at the most 50 cm. For instance, if in the original seesaw experiment we asked at what distance from the fulcrum should a 20-gm mass be placed to balance the 200 gm at 6 cm, we might do this: Let d be the distance; then

$$20 \text{ gm} \times d \text{ cm} = 200 \text{ gm} \times 6 \text{ cm} = 1200 \text{ gm} \times \text{cm}.$$

It seems that $d = 60$ cm. However, this answer is clearly nonsense. It is impossible to balance a 20-gm mass on a meter stick against a 200-gm mass placed at 6 cm from the fulcrum. The mathematics gave a nonsensical answer because we did not specify the domain of d.

In summary, a number sentence is

(a) an equation, if the number phrases are connected by the symbol "=", meaning equality;

(b) an inequality, if the number phrases are connected by any of the symbols, $\neq$, $>$, $<$; these symbols are verbalized "is not equal to", "is greater than", "is less than".

The set of numbers which make an open sentence true is called the truth set or solution set of the open sentence. To solve an open sentence means to find its entire set of solutions. The set of solutions of an open sentence may contain one member, or it may contain several members.

When the set of solutions of an open sentence has been found, we say that we have <u>solved the problem</u>.

Exercise 8

1. In the following problems assume that the domain of the variable is the set of all real numbers. Use your knowledge of arithmetic to find the solution set for each of the open sentences.

   (a) $x + 3 = 5$

   (b) $y + 3 > 5$

   (c) $4x = 12$

   (d) $4x \neq 12$

   (e) $\frac{n}{6} = 2$

   (f) $b + 8 < 10$

2. Replace the box with a number that will make the sentence true.

   (a) $\square + 3 = 12$

   (b) $a + \square = 8$

   (c) $3 \times \square + 2 = 23$

   (d) $4 \times \square = 20$

3. In each of the following examples, select those elements of the domain which make the open sentence true:

   (a) $x + 2 = 12$ $\{8, 4, 6, 10\}$ is the domain of x

   (b) $3x = 12$ $\{6, 2, 4\}$ is the domain of x

   (c) $16 - y = 10$ $\{8, 10, 6\}$ is the domain of y

   (d) $x^2 + 4 = 8$ $\{0, 2, 4\}$ is the domain of x

4. Let n represent the number of people that go to the local movie on Saturday night. What is the domain of n? If all tickets cost $1.35 each, and the total collection for one night is $235.25, how many people bought tickets?

5. Let g represent the number of gallons of gasoline you buy at the filling station. What is the domain of g? If each gallon costs 30¢ and you pay $2.76, how many gallons did you buy?

6. Let p represent the number of people who go to a dance at which only 30 couples are admitted. What is the domain of p? If each couple must be accompanied by a chaperone what is the domain of p?

1.11 Equations and Inequalities

The previous section dealt with open sentences. These sentences included relations between quantities which were equal, unequal, one greater or less than the other. Because of its frequency of use, the class of relations that are equal are called by a special name, "equations". Word phrases connected by the word phrase " = " state this kind of relation. Therefore, sentences stating equality between numbers or quantities are called equations.

The relation between the values obtained in one of the Balanced Meter Stick experiment was

$$120 \times d = 200 \times 6 \,.$$

The value d was obtained experimentally by sliding the 120 gm mass to bring the meter stick in balance. The value of d was found to be 10 cm. This value of d is the solution of the equation

$$120\ d = 200 \times 6 \,.$$

If the $\neq$, $>$, or $<$ relation connects the word phrases, the sentence is called an inequality. Such a relation occurs in the preceding experiment if the meter stick is not balanced. If the 120-gm mass is placed at a distance greater than 10 cm from the fulcrum, the relationship can be described as

$$120 \times d > 1200 \,.$$

Likewise, if the mass is placed closer than 10 cm, the relation becomes

$$120 \times d < 1200 \,.$$

Both relations are described by the statement

$$120 \times d \neq 1200 \,.$$

These are examples of inequalities. The important thing to notice is that "any statement which indicates that one number or quantity is not equal to another is called an <u>inequality</u>".

Exercise 9

Express in equation form the following :

1. Assume the cost of gasoline is 32¢ per gallon, and C represents the total cost of gasoline in cents. Write an equation for the total cost of n gallons of gasoline.

2. Write an equation for the cost d in <u>dollars</u> of n gallons of gasoline at 32¢ per gallon.

Write in symbolic form the following statements:

3. The diameter (D) of a circle equals twice the radius (r).

4. The perimeter (P) of a triangle equals the sum of its sides (a, b and c).

5. Which of the following sentences are true and which are false?

(a) $5 + (8 + 3) = (5 + 8) + 3$

(b) $6 + 4 \neq 2(4 + 1)$

(c) $5 + 2 = 3 + 4$

(d) $3.5 - 2.9 \not< 2.3$

(e) $8(3) \neq 3(8)$

(f) $\frac{1}{3} + \frac{1}{4} > \frac{1}{2}$

6. Write five true sentences involving each of the symbols,

$$<, >, \not<, \not>, \neq$$

7. Write five false sentences involving each of the symbols in Problem 2.

8. Put a numeral in place of the symbol $\square$ so that the sentence in each case will be true.

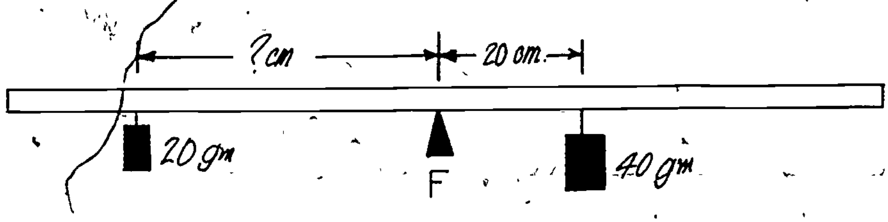
(a) $\square + 1 = 7$

(b) $\square - 3 \neq 5$

(c) $3 \times \square = 12$

(d) $6 \div \square < 2$

9. How far from the fulcrum should you place a 20-gm weight on the left side to balance a 40-gm weight of 20 cm from the fulcrum on the right side of the stick? (See illustration.)



10. How far from the fulcrum should you place a 20-gram mass on the left side of the stick to get the following inequality:

$20 \text{ gm} \times ? \text{ cm} < 40 \text{ gm} \times 2 \text{ cm}$?

To get $20 \text{ gm} \times ? \text{ cm} > 40 \text{ gm} \times 2 \text{ cm}$?

Can you get more than one answer?

1.12 Finding Unknown Masses by Experiment

The balanced meter stick can be used in performing other experiments.

Recall the rule that was obtained with the seesaw experiment. If the product of mass and distance on one side of the fulcrum equals the product of mass and distance on the other side, the meter stick is in balance. This rule can be used to measure the mass of any object, for example, a piece of rock.

Start by setting up the meter-stick instrument just as was done in the seesaw experiment. (See Figure 7.)

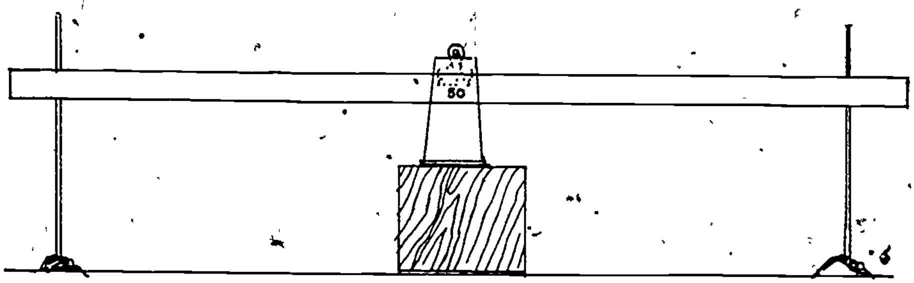


Figure 7

Attach a piece of string to a small rock so that it can be hung on the paper clip hooks. Have the standard masses at hand. Time can be saved if a standard mass with approximately the mass of the rock is selected. Hold the rock in one hand and a standard mass in the other. Select a standard mass which is approximately the mass of the rock.

Hang the rock of unknown mass at a convenient distance from the fulcrum on the left side. Use any convenient distance such as 20 cm. Then place the selected standard mass on the other side at about the same distance from the fulcrum. (See Figure 8.)

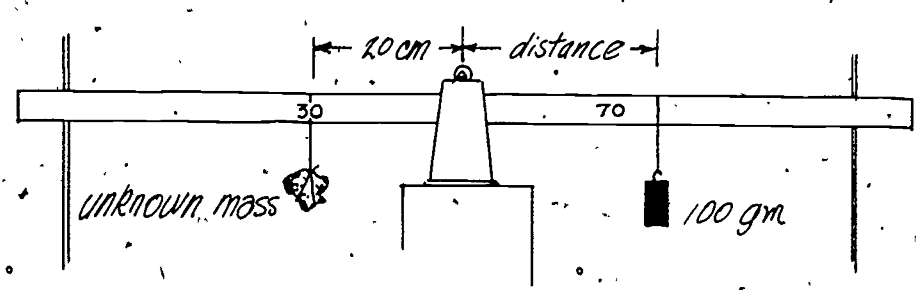


Figure 8

Did you get a balance?

Follow the procedure used in the seesaw experiment. Slide the standard mass closer to or further away from the fulcrum until you get a balance. Then read the distance to the nearest cm between the standard mass and the fulcrum. Write it down on a sheet of paper. Suppose it turned out to be 18 cm and the standard mass 100 gm. If the rock used is not exactly the same as your partners', the distance you read off will also be different.

The next problem is to find the magnitude of the mass of the rock. Actually, your balanced meter stick is an excellent illustration of a physical model of equality. Whenever the meter stick balances, the product of the mass on one side of the stick and its distance from the fulcrum must be equal to the product of the mass and its distance from the fulcrum on the other side, regardless of the masses used on either side. This physical model of equality can be described mathematically by an equation. In this case, the equation is

$$m \times 20 \text{ cm} = 100 \text{ gm} \times 18 \text{ cm} .$$

"m" is a symbol that represents the mass of the object. What is the value of m ?

If

$$m \times 20 \text{ cm} = 100 \text{ gm} \times 18 \text{ cm}$$

then

$$\frac{m \times 20 \text{ cm}}{20 \text{ cm}} = \frac{100 \text{ gm} \times 18 \text{ cm}}{20 \text{ cm}}$$

If two quantities are equal, they can be divided by the same magnitude and the quotients are still equal. For instance, if the mass of 40 golf balls is equal to the mass of 30 tennis balls, both masses can be divided by 5 and the quotients remain equal: the weight of 8 golf balls is equal to the weight of 6 tennis balls. By the same token,

if $m \times 20$ cm = 100 gm $\times$ 18 cm, then

$$\frac{m \times 20 \text{ cm}}{20 \text{ cm}} = \frac{100 \text{ gm} \times 18 \text{ cm}}{20 \text{ cm}}$$

$$m \times 1 = \frac{100 \text{ gm}}{20} \times 18$$

$$m = 5 \text{ gm} \times 18 .$$

$$m \doteq 90 \text{ gm}$$

Simplify both sides. Any number or magnitude divided by itself is 1. $\frac{20}{20} \times \frac{\text{cm}}{\text{cm}} = 1$, or $\frac{5}{5} = 1$, etc. $1 \times m = m$

The mass of the object is then 90 grams. To check that this answer is correct, substitute 90 grams for m in the equation.

$$m \times 20 \text{ cm} = 100 \text{ gm} \times 18 \text{ cm}$$

then,

$$90 \text{ gm} \times 20 \text{ cm} = 100 \text{ gm} \times 18 \text{ cm}$$
$$1800 \text{ gm} \times \text{cm} = 1800 \text{ gm} \times \text{cm}$$

This shows that m = 90 gm is the solution of the equation. If these values are used in the seesaw experiment, the mass of 90 grams at 20 cm from the fulcrum would balance the mass of 100 grams at 18 cm from the fulcrum. Could any other answer except 90 grams fulfill this condition?

Suppose we want to find the mass measure of a piece of rock. To simplify the arithmetic involved, hang the rock of unknown mass at 10 cm from the fulcrum and use a 100-gram sliding mass on the other side.

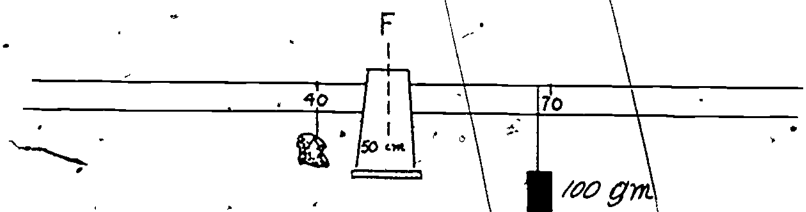


Figure 9

Then the sliding mass may be moved back and forth until the stick settles in a horizontal position. Suppose the distance from the fulcrum measures 6 cm.

Set up the equation: $m \times 10 \text{ cm} = 100 \text{ gm} \times 6 \text{ cm}$

where m is the mass measure of the rock.

Divide both sides of the equation by 10 cm $\quad \dfrac{m \times 10 \text{ cm}}{10 \text{ cm}} = \dfrac{100 \text{ gm} \times 6 \text{ cm}}{10 \text{ cm}}$

Using the associative law of multiplication $\quad m \dfrac{10 \text{ cm}}{10 \text{ cm}} = \left(\dfrac{100 \text{ gm}}{10}\right) \dfrac{6 \text{ cm}}{\text{cm}}$

$$1 m = 10 \times 6 \text{ gm}$$
$$m = 60 \text{ gm}$$

The mass measure of the rock is 60 gm.

Exercise 10

Find the mass of a stone by using the meter stick instrument. Use a procedure similar to that just described. Place the object with the unknown mass on the right side of the stick at 10 cm from the fulcrum and hang the 100-gm sliding mass on the left side. Read off the distance when the stick is in balance to the nearest cm. Determine the mass of the object in grams. Repeat this procedure using three other unknown masses.

1.13 Multiplicative Inverse

Suppose we have the problem of finding the mass of another object. After comparing its mass with that of the standard mass, it is found that the mass of the object is approximately the same as that of the 100-gm standard mass. It is necessary to find the actual mass of the object.

Place the object at 7.5 or $\frac{15}{2}$ cm on the left side of the fulcrum and the 100 gm at about the same distance on the other side of the fulcrum. (See Figure 10.)

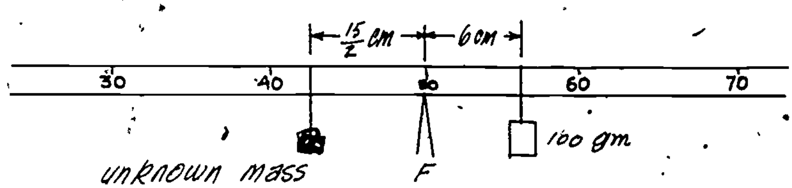


Figure 10

After sliding the 100-gm mass back and forth, suppose we get a balance at 6 cm.

Let X represent the mass of the object. Then, using our rule, set up an equation as follows:

$$\frac{15}{2} \text{ cm} \times X = 6 \text{ cm} \times 100 \text{ gm}$$

In this problem we are supposed to divide both sides of the equation by $\frac{15}{2}$. This may seem to be a complicated computation. However, mathematicians have a better way of solving this type of equation by using the multiplicative inverse. Let us consider this concept.

Perform the multiplication of the indicated numbers:

$$\frac{1}{2} \times 2 =$$

$$\frac{2}{3} \times \frac{3}{2} =$$

$$5 \times \frac{1}{5} =$$

$$\frac{7}{8} \times \frac{8}{7} =$$

Do you see any pattern in performing the multiplication?

Look at these problems closely. Notice that in each instance we multiplied the number by another number such that the product is always 1. This pattern leads us to another property of numbers, namely, "For every number, except 0, there is another number called the <u>multiplicative inverse</u>, such that the product of these numbers is always 1." For example,

$$\frac{1}{2} \times 2 = 1;$$

hence, 2 is the multiplicative inverse of $\frac{1}{2}$. In the second problem,

$$\frac{2}{3} \times \frac{3}{2} = 1.$$

Another name for the multiplicative inverse of a number is the <u>reciprocal</u> of a number; for example, instead of saying that 8 is the multiplicative inverse of $\frac{1}{8}$, we can say that 8 is the reciprocal of $\frac{1}{8}$. Also, $\frac{3}{2}$ is the reciprocal of $\frac{2}{3}$. Finally, it is true that the product of reciprocals is one. The reciprocal of 1 is 1 because $1 \times 1 = 1$.

What is the multiplicative inverse of 0? Do you know of any number that multiplied by 0 equals 1? Let's see:

$$5 \times 0 = 0$$
$$0 \times 200 = 0$$

In fact, we know that the product of 0 and any number is 0. Therefore, 0 has no multiplicative inverse.

<u>Examples</u>. State the multiplicative inverse in each of the following:

$\_\_\_\_ \times 7 = 1$	Answer:	$\frac{1}{7}$
$\frac{9}{8} \times \_\_\_\_ = 1$	Answer:	$\frac{8}{9}$
$a \times \_\_\_\_ = 1$	Answer:	$\frac{1}{a}$ if a is not 0

In summary, every number except 0 has an inverse with respect to multiplication. We call this the <u>multiplicative inverse</u>. In the next section we will show how the multiplicative inverse can be used to solve an equation.

Exercise 11

Find the multiplicative inverse of each of the following numbers:

1. (a) 17
   (b) $\frac{8}{7}$
   (c) $\frac{4}{5}$
   (d) 1
   (e) .6
   (f) X
   (g) $\frac{7}{3}$
   (h) 4
   (i) $\frac{\frac{1}{2}}{3}$
   (j) a + 1
   (k) $5\frac{1}{2}$
   (l) $\frac{1}{(ice)^3}$

1.14 Solving Equations

The problem in the previous section stated

$$\frac{15}{2}\text{ cm} \times X = 6\text{ cm} \times 100\text{ gm} .$$

This equation is solved when X stands by itself on the left-hand side of the equation. To obtain this, multiply $\frac{15}{2}$ cm by $\frac{2}{15\text{ cm}}$, the multiplicative inverse of $\frac{15}{2}$ cm ;

$$\left(\frac{2}{15\text{ cm}}\right)\frac{15\text{ cm}}{2}\ X = 1X.$$

To keep our equality, we must multiply the other side of the equation by $\frac{2}{15\text{ cm}}$ also.

The equation then becomes

$$\left(\frac{2}{15\text{ cm}}\right)\frac{15\text{ cm}}{2}\ X = \left(\frac{2}{15\text{ cm}}\right)6\text{ cm} \times 100\text{ gm}$$

$$\left(\frac{2}{15\text{ cm}} \times \frac{15\text{ cm}}{2}\right) X = \left(\frac{2}{15\text{ cm}} \times 6\text{ cm}\right) 100\text{ gm}$$

$$1\ X = \frac{12 \times 100\text{ gm}}{15}$$

$$X = \frac{1200}{15}\text{ gm}$$

$$X = 80\text{ gm}$$

This means that 80 gm at $\frac{15}{2}$ cm from the fulcrum balances 100 gm placed at 6 cm from the fulcrum. Check it on your meter stick.

Examples: Find the solution of each of the following open sentences; then check your answer.

Illustrative Example:

$$\frac{3}{4}X = 60$$
$$\left(\frac{4}{3}\right)\frac{3}{4}X = \left(\frac{4}{3}\right)60$$
$$\left(\frac{4}{3} \times \frac{3}{4}\right)X = \frac{240}{3}$$
$$1X = 80$$
$$X = 80$$

Multiply both sides of the equation by $\frac{4}{3}$, the multiplication inverse of $\frac{3}{4}$.

Check: If X = 80, then the left member is $\frac{3}{4}(80) = 60$, and the right member is 60. Therefore, $\frac{3}{4}(80) = 60$ is a true sentence, and the solution is 80.

Exercise 12

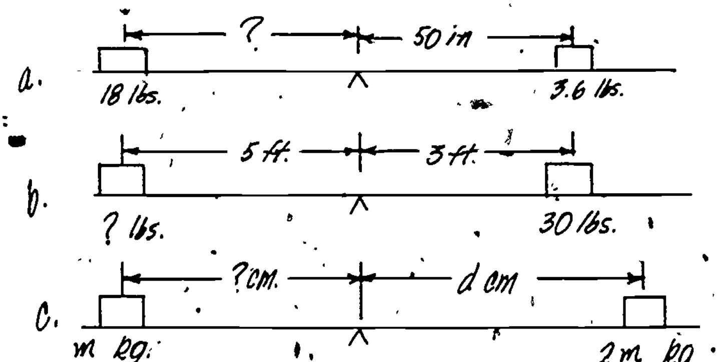
1. Solve the following by use of the multiplicative inverse.

(a) $12x = 6$
(b) $7x = 14$
(c) $\frac{8}{7}x = 56$
(d) $15 = \frac{5}{3}y$
(e) $5y = 2$
(f) $\frac{7}{3}x = 1$
(g) $\frac{2}{3}a = \frac{2}{3}$
(h) $10x = \frac{8}{5}$
(i) $2.3y = 4.6$

2. Translate each of the following sentences into symbols and then solve the equation for the unknown.

(a) The number x multiplied by 5 is equal to 30.
(b) When a number y is divided by 4 the quotient is 9.
(c) The product of $\frac{2}{7}$ and the number a is 28.
(d) Jane bought x stamps for 3¢ each. How many stamps did she buy if she paid 60¢ altogether?
(e) How old is Susan if 9 times her age is 63?

3. Find the missing values in each case:



4. Do you suppose a 90-pound girl could ever lift a 1000-pound box? Justify your answer.

5. A child whose weight is 60 pounds asked his father, whose weight is 180 pounds, to ride a seesaw with him. Where should the father sit to balance the child if she sits 6 feet from fulcrum?

6. A bar 6 feet long is being used as a lever to lift a stone. What is the weight of the stone if a boy weighing 100 pounds pushing down on one end of the bar which is 4 feet from the fulcrum just balances the stone on the other end?

1.15 Summary

The experiments in this chapter provided data from which a relation could be determined for balancing a seesaw. To find this relation, it was necessary to learn about number phrases, word phrases and verb phrases. The word phrases and verb phrases gave us an open sentence which was the mathematical expression of the experimental data. It was found that the open sentence stating the condition of balance of the seesaw was an equality between two quantities and, therefore, an equation. When the seesaw was not in balance, the open sentence was an inequality.

When the mass on the seesaw was an unknown, the truth set or solution set of the equation was not obvious. It was necessary to solve an equation to determine the value of the mass. The properties of the multiplicative inverse and equality were used to find the solution of the equation.

Chapter 2

AN EXPERIMENTAL APPROACH TO LINEAR FUNCTIONS

2.1 The Loaded Beam

At this time we will investigate the bending of a beam as the load upon it is changed. We will use the same beam throughout the experiment. It will be clamped in the same position and always loaded from the same point. By fixing the beam in this manner, we are in a position to study the relationship between the bending of the beam and the amount of load. This prevents other factors from entering directly into the experiment.

A 15-inch flexible ruler may be clamped to a desk with a "C-clamp" and used as a beam. There should be a small hole in the ruler about one inch from the free end. Fasten a piece of strong thread to the ruler and pass the free end through the hole. The thread will be used for attaching loads to the beam. To measure the bending of the beam, we will simply record the changing position of the free end of the beam as the load is changed. You may find that some form of a pointer arrangement, such as a straight pin fastened to the free end, will be helpful.

Support a meter stick perpendicular to the floor so that the position of the end of the beam can be read on the scale as the load changes. The smaller numbers on the meter stick should be at the top. (Figure 1.)

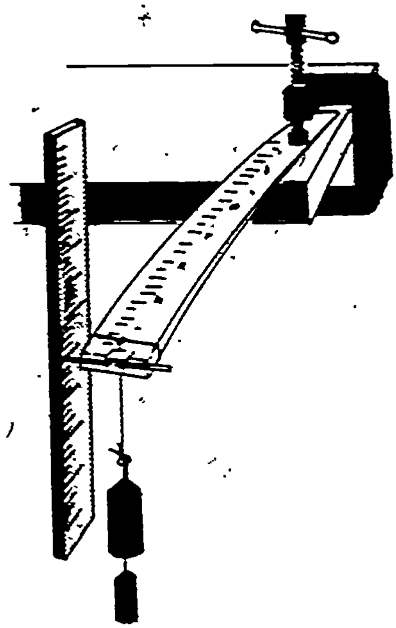


Figure 1

First take a reading of the position of the beam with no load attached. Now hang a 30-gram mass from the load point and take a new reading of the position of the end of the beam. Continue in this way, adding 30 grams each time, until you have at least ten readings. Be very careful in reading the position of the free end of the beam. Always try to "sight" along the pointer in the same way. Make your position reading to the nearest tenth of a centimeter.

You should record your data in an orderly fashion. Along with the load values and the position readings you should record such things as the type of beam used and its length, i.e., that part which extends outward from the table top to the load point. In recording the position of the end of the beam that is associated with each load, a tabular arrangement will have the most meaning. For example, you could now label two columns for data, one column with the heading, "Load ℓ in gm" and the other, "Position (p) in cm". If you wish to make more than one trial run on loading the beam, you will need more than one column in your table for the position of the pointer. (Table 1.)

THE LOADED BEAM EXPERIMENT

Type of beam __________ Length of beam __________

Load ℓ (grams)	Trial 1 Position p (centimeters)	Trial 2 Position p (centimeters)	Trial 3 Position p (centimeters)
0			
30			
60			
.			
.			
.			

Table 1

Now go back and run through the experiment again. This time start with a 60-gram mass and continue by adding 10 grams each time until you have at least ten readings. Record these readings in a new data table and put this table aside for later reference in Section 2.4.

2.2 Graphing the Experimental Points

If we now examine the data we see that our table pairs up a certain value for the position (p) of the end of the beam with a certain value (ℓ) of the load. The table shows that there is a certain relationship between the load and the position of the end of the beam. The value we obtain for the position of the end of the beam <u>depends</u> on the load that we hang on the end of the beam. In other words, our data is a set of <u>ordered pairs</u>. As we have seen before, we can represent ordered pairs of numbers by using coordinate paper. In doing the experiment, we have decided what loads to hang from the beam. The resulting position of the end of the beam has <u>depended</u> on this load. The general practice is to make the first element of the ordered pair the measure that we controlled. Thus, for this experiment, the first element in the ordered pairs will be the load value, and the second element will be the position reading associated with this load value. Our ordered pairs become (ℓ, p) pairs. It will be helpful to label the horizontal axis the "ℓ" axis (load) and the vertical axis the "p" axis (position).

A sample of the ordered pairs which you might get from this experiment could look like this: (0, 20.0) (30, 20.5) (150, 22.0) (300, 24.0). We went from an unloaded beam to a beam supporting a load of 300 grams. At the same time, the pointer only moved from the 20 cm mark to the 24 cm mark.

We are going to use the graph of these ordered pairs to help us make decisions about the behavior of the bending beam. In order for the graph to give a good picture of the actual experiment, appropriate scales should be chosen for both axes. In this particular case, the horizontal scale should go from 0 to 300 while the vertical scale extends from 0 to 25. Once the data is plotted, you probably will have something which looks like Figure 2.

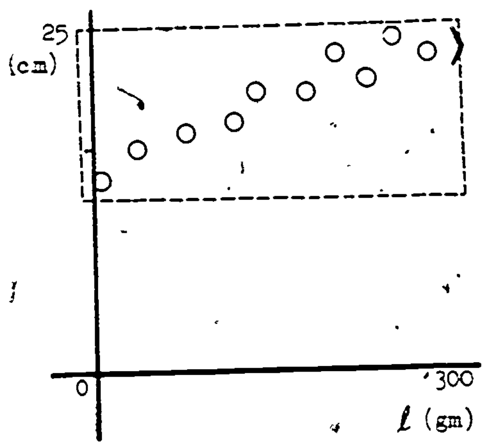


Figure 2

The only section of the coordinate plane which is of interest to us is that enclosed by the dotted line in Figure 2. We can make even greater use of the coordinate paper if we use only that part of the graph enclosed by the dotted lines.

The horizontal and vertical lines drawn on the graph paper are called <u>scales</u>. These scales are not necessarily the axes of the coordinate plane. In Figure 3, the horizontal scale is a line above and parallel to the horizontal axis. The vertical scale, however, is part of the vertical axis. <u>The intersection of these two scales is not the origin</u> but the point whose coordinates are (0, 20).

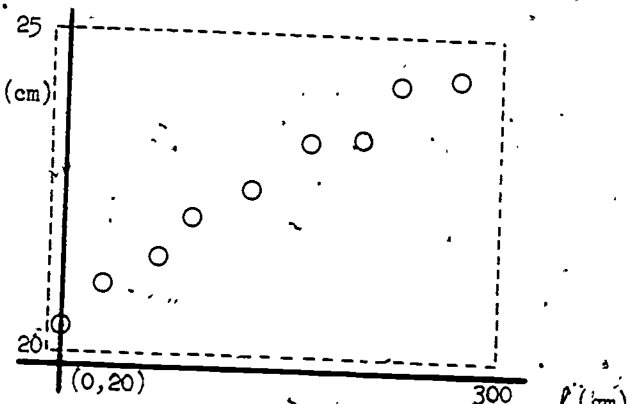


Figure 3

Whenever you plot data, you should follow a method similar to the one just discussed. It is not necessary to draw two or more separate graphs to do so, of course. <u>Examine</u> your data to decide upon a good scale to use for your graph. Decide just where the graph falls in relation to the entire coordinate plane. Use only this part of the entire plane for your graph of the data.

2.3 Connecting Plotted Points

Once the scales have been set and the data plotted, we have the problem of interpreting the meaning of the space between these points. Examine the set of points you have just plotted. Does their arrangement suggest anything to you? Suppose that during the experiment more load-position readings were made. If we had increased the load by one-tenth of a gram each time, instead of by 30 grams, we would still find a new position reading for each load. Actually reading the change in position for such a small load change may be difficult, but the fact that there will be a change should be obvious. We now have to decide how the position of the end of the beam would vary with changing load. The variation is probably quite regular each time the load is increased. Let us guess that the position would change only half as much if the increase in load were 15 grams instead of 30 grams. There is no reason to suppose that a regular change of position would not occur between these points. Our first guess, for a model of the behavior of the end of the beam with changing load, would then be to join our experimental points with straight line segments. This procedure will give us something like the graph shown in Figure 4.

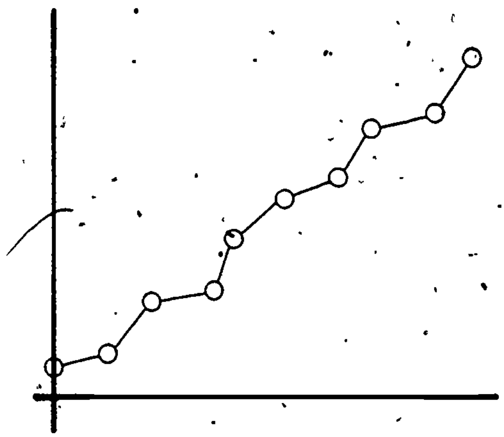


Figure 4

2.4 The Best Line

This method of joining our experimental points is perhaps not the best model we can construct. When we say that the beam behaves exactly like our experimental points, we are saying that our readings are exact. Can you think of any errors in your data? This graph is also the result of a single trial of the experiment. The errors which can occur may be great enough to make this model meaningless. For this reason, scientists and mathematicians do not like to draw conclusions on the results of a single trial.

If we were to repeat the experiment a number of times and graph each set of data on the same sheet of coordinate paper, you would probably arrive at a figure like that shown in Figure 5. This figure shows us something about our ability to reproduce the experiment. (Do we obtain "about" the same ordered pairs a second and third time?) It also suggests that the "spread" of the plotted points may be due to certain inaccuracies involved in the measurements, either in the load value, the position value, or both. Perhaps

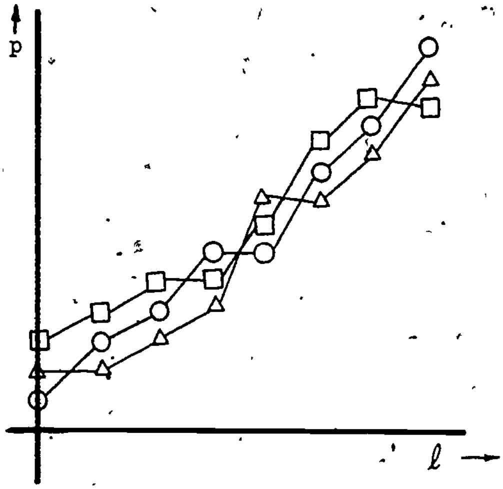


Figure 5

the plotted "points" should not be points at all, but small areas. This last statement illustrates that measuring instruments are not perfect. In fact, instruments designed to measure the same thing, may differ among themselves. The new ordered pairs that we get when we "reproduce the experiment" point out that a single person may get different results even when using the same instruments for making repeated measurements.

We can sum up the preceding discussion by saying that there are at least two classes of error that must be considered when we make any sort of measurements, namely, instrumental errors and human errors. We can cut down the magnitude of these errors by making our instruments as accurate as possible, and then by using them as carefully as we can. However, we cannot eliminate the errors completely. Therefore, we should keep their existence in mind as we interpret the results of our measurements. In this way, we can usually see what fundamental relations there are between quantities, in spite of unavoidable errors.

What we are calling <u>fundamental relations</u> are the results we would predict if we could be sure no errors in measurement had been introduced. Let us call an experiment which introduces no error an <u>ideal experiment</u>. We are led to the conclusion that the results of an actual experiment, and the results of an ideal experiment using perfect equipment and exact measurements are two entirely different situations. In our experiment we have a relationship between position and load in the form of a data table and in the form of a graph. What we desire now is a "physical model" to explain the behavior of the beam. The data from each trial, and the <u>braid</u> arrangement of the data, as shown in Figure 5, seem to suggest a straight line. You may not be able to find a straight line which will connect all the points for any one trial, but with a little practice, you should be able to find a line which seems to "best" represent all of the data. This "best straight line" will be our <u>physical model</u> of a relation we have "guessed". This line represents our model of an ideal experiment.

Once we have decided to depart from the experimental "facts" and draw a single straight line to represent our data, we have a graph similar to that in Figure 6. This graph gives a pictorial relation of load and position. Our problem now is to find a mathematical representation of this relation. We now have a relation between load and position in terms of recorded data and a graph of this data. We have also formed a physical model to represent an ideal experiment suggested by this data. We now want to obtain a mathematical model which will describe the position of the end of the beam in terms of load. This is our third step in the analysis of the experiment.

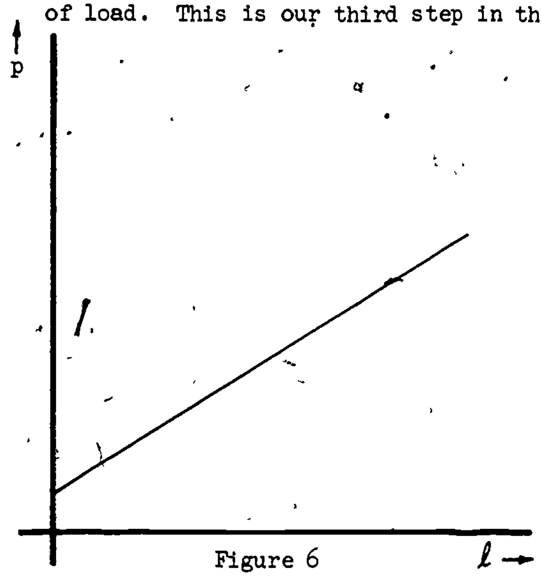


Figure 6

In comparing the physical model you have constructed with those of other students, you may notice that different groups of students will have graphs which start at different points, or differ in their "steepness", or both. Can you think of any reason for these differences? Check your data. Recall how much of the beam extended out from the table edges. Did each student have the beam extend out from the table edge by the same amount? Was the beam you used exactly like the beam used by other students? What was your "zero" reading, i.e., your reading when there was no load on the beam?

Exercise 1

1. Referring to your final graph of load-position pairs for the loaded beam, is the horizontal scale drawn along the horizontal axis? Is the vertical scale drawn along the vertical axis?

2. Give a good reason why coordinate axes do not always appear on your coordinate paper.

3. On a sheet of coordinate paper, draw horizontal and vertical axes with the origin at the lower left-hand corner. Number the horizontal axis from 0 to 200. Number the vertical axis from 0 to 10. Plot the following set of ordered pairs relating temperature and time:

   {(160, 8.0), (170, 8.6), (180, 9.1), (190, 9.4), (200, 9.9)}

4. Make a new graph of the points of Exercise 3 in such a way that the graph nearly "fills" the coordinate paper. Label both the horizontal and vertical scales.

5. Draw your "best" straight line through the points plotted in Exercise 4. Why do some of the points fall off the line?
If the horizontal coordinates are the temperatures of an iron rod in degrees Centigrade, and the vertical coordinates are the corresponding times in minutes, is the drawing of the line justified?

6. Referring to the exercise above, what is the time corresponding to a temperature of 165°C? What is the temperature corresponding to a time of 9.3 minutes?

7. For each of the following, plot the points whose coordinates are given, and then draw what you judge to be the best line. Read the y-value of the point at which your line crosses the y-axis and compare the results with your classmates.

(a) (15, 17.5), (25, 20.0), (45, 27.5), (55. 30.0)
(75, 37.5), (80, 40.0), (100, 45.0), (120, 50.0),
(125, 52.5), (135, 57.5), (160, 65.0)

(b) (0.2, 12.5), (0.4, 12.0), (1.0, 11.0), (1.4, 10.0),
(1.8, 9.5), (2.8, 7.5), (3.6, 6.0), (4.2, 4.5),
(5.2, 3.0), (5.8, 1.5)

(c) (0, 0), (1, 5), (2, 9), (5, 18), (6, 22), (8, 29),
(9, 34), (10, 37)

(d) (150, 33), (300, 31), (450, 31), (600, 32), (750, 31), (900, 34),
(1050, 33), (1250, 32), (1300, 30), (1500, 33), (1650, 32)

2.5 Slope

You may recall from your study of the number line that the distance from one point to another is the coordinate of the one point minus the coordinate of the other. For example, the distance between the points whose coordinates are 2 and 7 is 7 - 2 or 5. If the points are not on the number line, but are points on the coordinate plane, the question of finding the distance between these

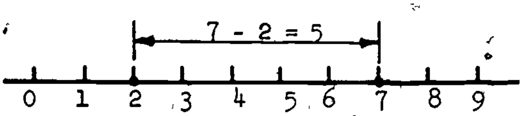


Figure 7

points becomes much more complicated. There are some cases, however, which are not too difficult to determine. If the straight line which connects the two points is either horizontal or vertical then the distance between the points is again only the matter of subtracting one coordinate from another.

We will first consider the case of a horizontal line. If every point of a line on the coordinate plane has the same second element, then we define this to be a horizontal line. The line illustrated in Figure 8 is an example of a horizontal line. What is the distance between the two points whose coordinates are (3, 2) and (7, 2)? Let us define the distance between two points on a horizontal line as the first element of one ordered pair subtracted from the first element of the other ordered pair; that is, 7 - 3. Therefore, in this example, the distance between the two points is 4.

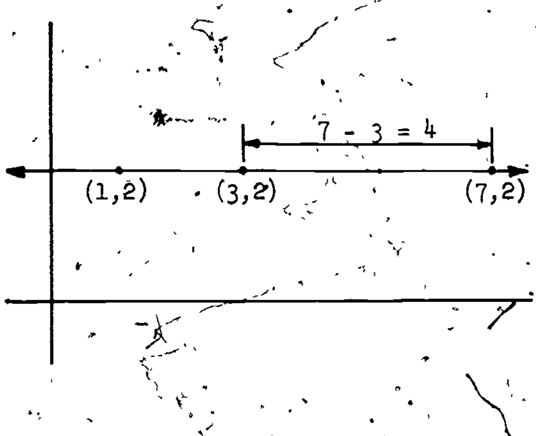


Figure 8

For a vertical line, we shall follow a similar procedure. If the ordered pairs describing the points of a line on the coordinate plane all have the same first element, then we define this line as a vertical line. The distance between any two points on a vertical line is the second element of one ordered pair subtracted from the second element of the other ordered pair. It follows, then, that the distance between the points whose coordinates are (3,1) and (3,5) is 5 - 1 or 4.

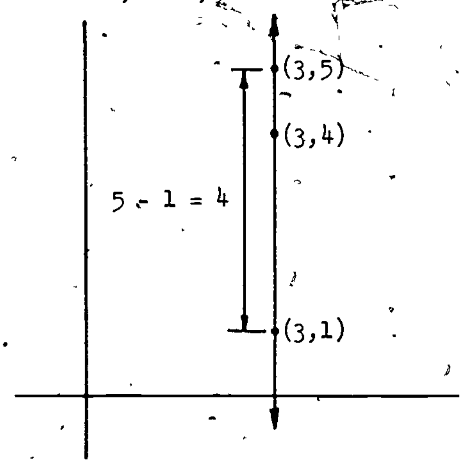


Figure 9

If two points have coordinates such that the first elements of each are different and the second elements are also different than the line drawn through these points is neither horizontal nor vertical. The ordered pairs (2,3) and (7,4) determine such a line. As we scan this line (Figure 10) from left to right, we notice that it slopes up. We might ask, at

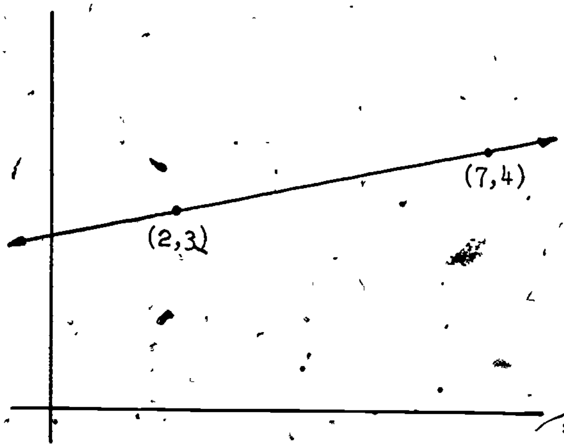


Figure 10

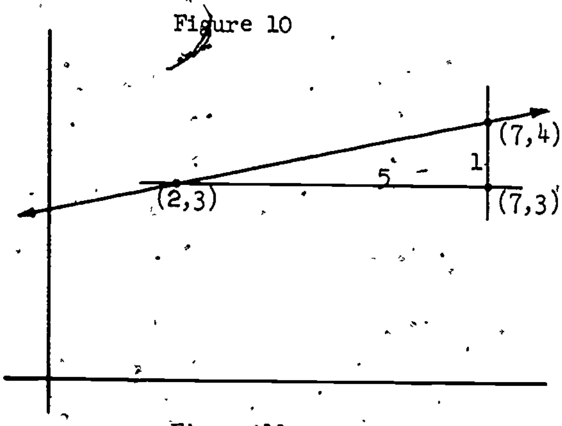


Figure 11

this time, if there is any way to compare the "steepness" of the slope of such lines which are neither horizontal nor vertical.

Before we actually answer this question, let us look at the line drawn in Figure 10. If we draw a horizontal line through the point whose coordinates are (2,3) and a vertical line through the other point (7,4), we have two new lines which intersect at a new point. This point is a point of a vertical line which passes through the point (7, 4). By definition of a vertical line, the horizontal coordinate of this new point is 7. The point is also a point of a horizontal line, which, by definition, must have a vertical coordinate of 3. Therefore, the coordinates of this new point are (7,3).

The distance, on the vertical line, between the points (7,4) and (7,3) is 4 - 3 = 1. This vertical distance is often referred to as "rise". The distance, on the horizontal line, between the points (2,3) and (3,3) is 7 - 2 = 5. This horizontal distance is referred to as "run". The ratio of the "rise" to the "run" is called the <u>slope of the line</u>. The slope of the line in this example is $\frac{1}{5}$.

For a straight line the "steepness" is the same along the total length of the line and the slope will be the same between any two points we might pick. The letter m is usually used for the slope. Thus, for a straight line we have

$$m = \frac{\text{rise}}{\text{run}} = \text{a constant.}$$

We note that in finding the rise we subtracted 3 from 4. These numbers were the second elements of the original ordered pairs which we used to find the line. The run was determined by subtracting 2 from 7. These numbers were the first elements of the original ordered pairs. From this it appears that it is not actually necessary to draw in the horizontal and vertical lines through the points in order to find the slope of the line.

The slope of a line through the points whose coordinates are (a,b) and (c,d), where the second point is to the right and up from the first point is $\frac{d - b}{c - a}$. As we scan this line from left to right, we find that it slopes up.

As an example of this procedure, suppose that two points have the coordinates (8,18) and (16,28). (Figure 12.) The rise will be 28 - 18 and the run 16 - 8. The slope of this line will then be

$$m = \frac{28 - 18}{16 - 8} = \frac{10}{8} = \frac{5}{4}.$$

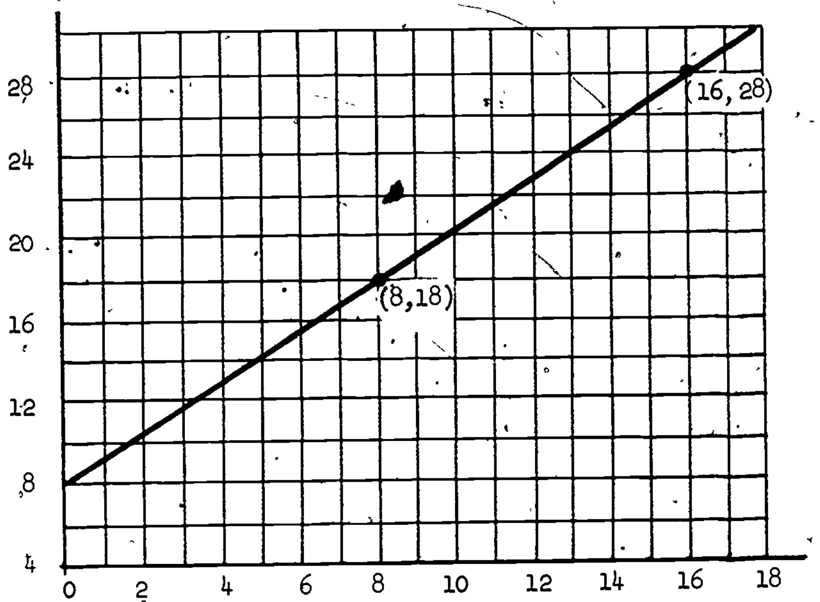


Figure 12

<u>Exercise 2</u>

1. Which of the following two ordered pairs determine a horizontal line, a vertical line and a line which is neither.

(a) (3, 2), (5, 2)
(b) (0, 0), (7, 0)
(c) (10, 4), (4, 10)
(d) (5, 6), (6, 7)
(e) (2, 8), (4, 8)
(f) (2,3), (2, 2)
(g) (561, 10), (562, 11)
(h) (3, 14), (6, 28)
(i) (9, 8), (9, 1)
(j) (0, 8), (0, 5)

2. For each of the following two ordered pairs state the rise and the run for the line determined by these points.

(a) (2, 5), (4, 8)
(b) (3, 9), (2, 1)
(c) (8.5, 7), (9, 9)
(d) (20, 10), (25, 17)
(e) (5, 3), (5, 986)
(f) (763, 763), (25, 25)
(g) (8, 7), (2, 5)
(h) (8, 10), (0, 10)
(i) (3.7, 12.6), (5.2, 13.1)
(j) $(\frac{3}{4}, \frac{5}{6})$, $(\frac{5}{4}, \frac{11}{6})$

3. Determine the slope of the line connecting the points in each part of Problem 2.

2.6 Equation of a Straight Line - Slope-Intercept Form

The starting point of the graph of the loaded beam relation may have differed from group to group. This is the point where the line intersects the vertical axis. Thus, the "horizontal" coordinate of this point will be zero. We know that the ratio of the rise to the run (the slope) will be the same for any two points on the line. If we know the value of the slope of a line and select the point at which the line intersects the vertical axis with coordinates (0, b) as the first point, then, for any arbitrary second point with coordinates (ℓ, p) we have

$$\frac{p - b}{\ell - 0} = m .$$

Since $\ell - 0$ is the same number as ℓ, we could rewrite this statement $\frac{p - b}{\ell} = m$. Multiplying both sides of this expression by ℓ we get

$$\ell \times \frac{p - b}{\ell} = m\ell$$

But $\frac{\ell}{\ell}$ is the same as 1, so we can again rewrite to get

$$p - b = m\ell$$

and finally

$$p = m\ell + b.$$

Every straight line, except a vertical line, can be given an equation of this form. The equation,

$$p = m\ell + b$$

was derived from our definition of slope and the statement that all portions of the line have the same slope. If we look more carefully at the derivation of this equation, you will recall that we began with two ordered pairs, one of which was of the form (0, b). This point has a special significance. This is a point on the vertical axis. Since we have already said that this line cannot be a vertical line, we know that it can cross the vertical axis in exactly one point whose coordinates are (0, b). This point is referred to as the intercept. Looking again at the equation in this form, we note that the factor m is the slope of the line and the term b is the intercept. Hence, this form of the equation of a straight line is called the "slope-intercept" form of the equation.

In Figure 12, we determined that the value of the slope of the line was $\frac{5}{4}$. From the figure we see that the coordinates of the vertical intercept are (0, 8) and

$$\frac{p - 8}{\ell - 0} = \frac{5}{4}$$

and the equation of this line is

$$p = \frac{5}{4}\ell + 8.$$

With this equation we can predict position of the pointer for any given load. What position would you predict for a load of 16? In this case,

$$p = \frac{5}{4}(16) + 8$$

or

$$p = 5(4) + 8 = 28.$$

The graph did have an ordered pair (16, 28) and we see that this equation does give us a method for finding ordered pairs that are the coordinates of points on the line.

If we refer back to the set of ordered pairs presented in Section 2.2, we can derive an equation for the graph of the best line determined by these ordered pairs. Graphing these ordered pairs {(0, 20.0), (30, 20.5), (150, 22.0), and (300, 24.0)} and drawing the "best line" would probably give a graph like that in Figure 13. The ordered pair (30, 20.5) falls just off the line and we might justify this on the basis that the second element of each ordered pair was measured to the nearest 0.5. In this case the intercept is 20 and the slope is $\frac{24 - 20}{300 - 0} = \frac{4}{300}$ or $\frac{1}{75}$. Hence, b = 20 and $m = \frac{1}{75}$ so the equation of our "best line" is $p = \frac{1}{75}\ell + 20$.

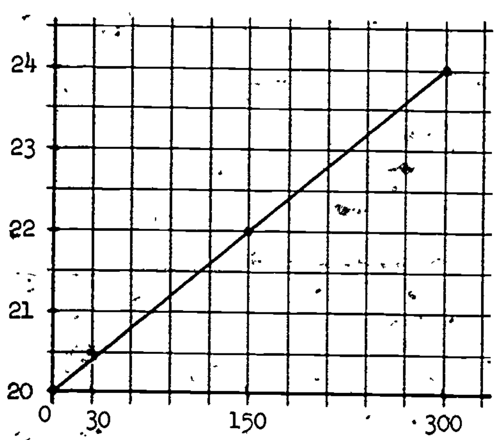


Figure 13

We now have a mathematical model which can be used. If we insert values for the load in the equation we can now calculate a corresponding position value. If we calculate position values for loads which were used in the experiment, we can find how closely our model agrees with our actual observations. Moreover, we can use this model to predict position values for loads which were not actually used in the experiment. Try this. Pick a load not previously used but in between the extreme values. Use your equation to predict the position of the end of the beam, and then find the position for this load experimentally. Do the predicted and observed values tend to agree?

Can we also use this equation to predict deflection readings for loads outside the range used? We have to be careful in using this process. The equation seems to give us values for the position of the end of the beam for upward bending, and for loads which far exceed the "breaking point" of our beam. Thus our equation must be limited to values of ℓ which are in the interval from 0 to 300.

We have now achieved the aim of the experiment. We have learned how to investigate the possible variables and how to isolate and observe those which are of particular interest to us. We found the relation between load and position both in terms of tabular data and a graph of the data. We then made a physical model of the experiment by representing the data as a "best" straight line. Finally, we found a mathematical representation of this physical model. In future experiments we will use these concepts again and develop new techniques, both experimental and mathematical, to help explain our physical surroundings.

Exercise 3

1. Tabulate the coordinates of the points P, Q and R, shown in the accompanying graph. Calculate the slope of line ℓ_1, using the points P and Q. Do the same for points P and R and again for points Q and R.

2. Referring to Figure 14, what are the slopes and vertical axis intercepts of lines ℓ_2, ℓ_3, and ℓ_4?

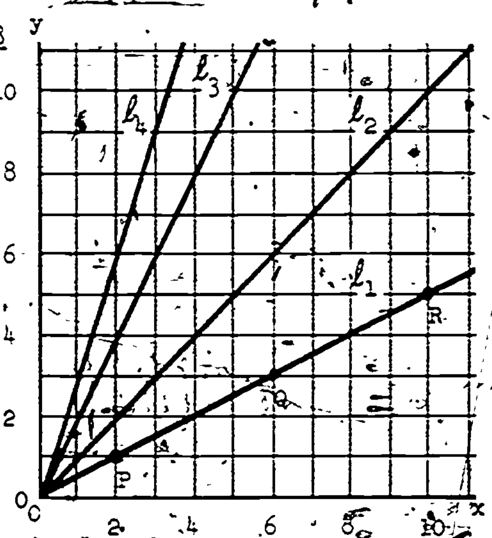


Figure 14

3. Find the slope and intercept of lines ℓ_5 and ℓ_6 (Figure 15).

4. What do you need to know about a line to distinguish it from any other line?

5. Write the equations for the lines ℓ_1, ℓ_2, ℓ_3, ℓ_4, ℓ_5, and ℓ_6 shown in Figures 14 and 15.

6. In the loaded beam experiment, what is the significance of the position axis intercept that you obtained? Would a different intercept have given you a different slope?

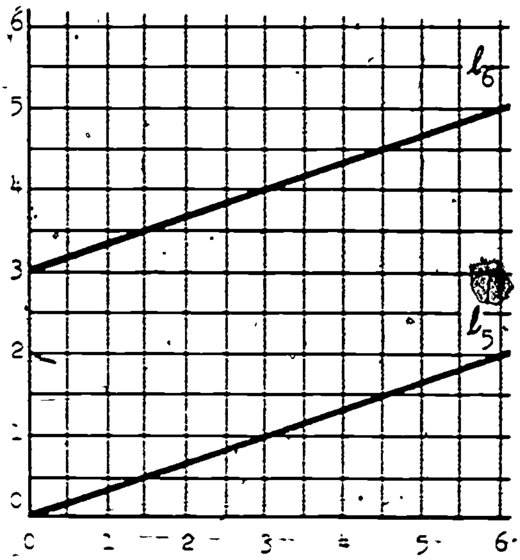
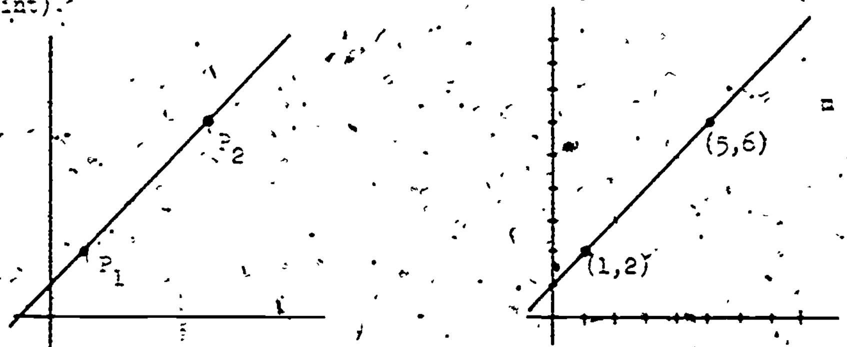


Figure 15

2.7 Graphing Linear Equations

We have seen that slope is a very important concept in discussing the mathematical description of a line. We have defined the slope of a line by using the coordinates of two distinct points on the line. The slope of a given line does not depend on the particular pair of points used to determine the line, nor on the relative position of these two points. The examples below review the various possibilities discussed here. Each of the examples shows a general situation and a specific example.

Example 1: P_2 (second point) is above and to the right of P_1 (first point).



$$m = \frac{6-2}{5-1} = \frac{4}{4} = 1$$

Figure 16

The slope is positive. The line rises as we proceed from left to right.

Example 2. P_1 and P_2 have the same vertical coördinate.

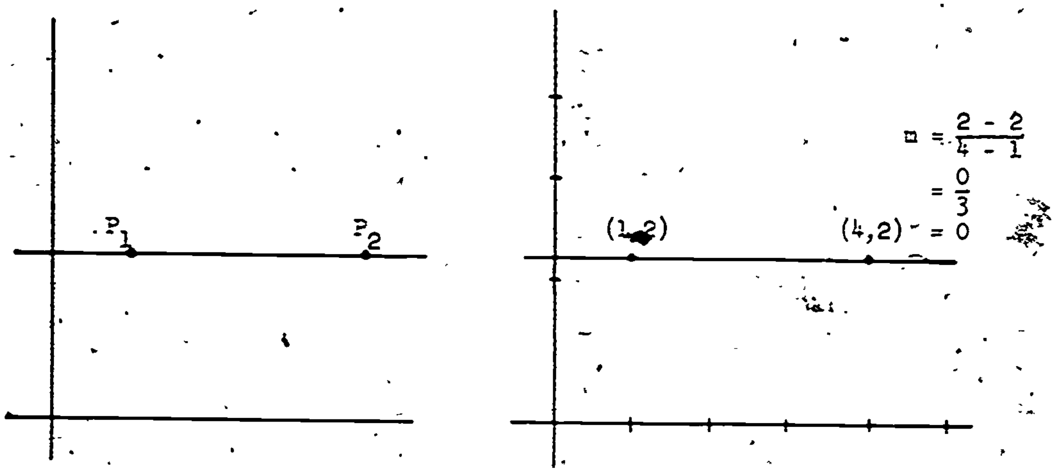


Figure 17

The slope is zero. The line is horizontal.

Example 3. P_1 and P_2 have the same horizontal coordinate.

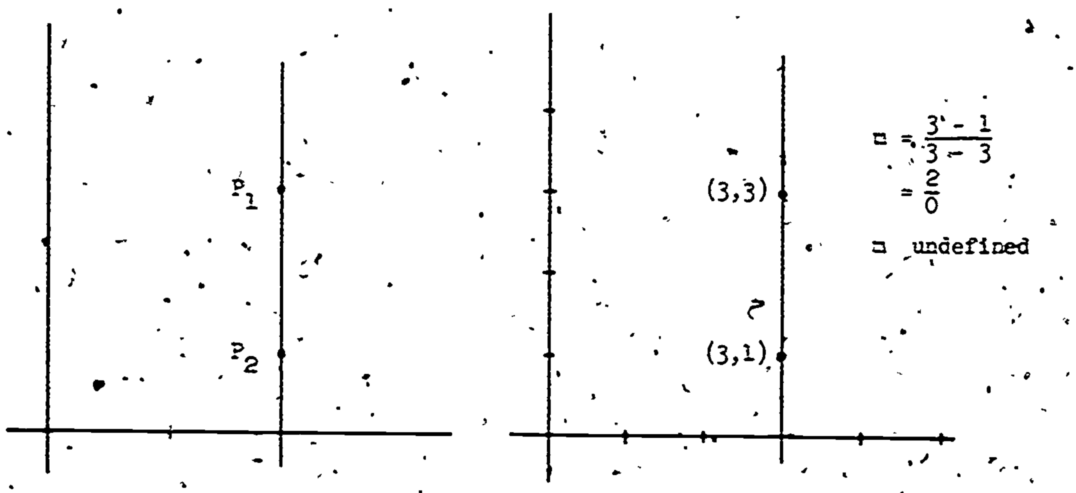


Figure 18

The slope is undefined. The line is vertical.

Our discussion of slope has ignored one general situation that may develop. That is when P_2 is below and to the right of P_1, as shown in Figure 19.

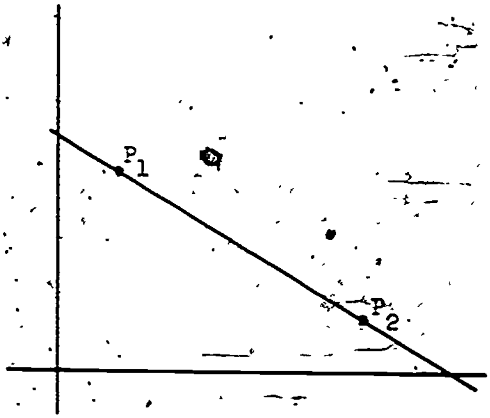


Figure 19.

We will reserve discussion of this situation for a later course.

We may summarize the preceding results as follows.

If $m > 0$, the line rises to the right.

If $m = 0$, the line is horizontal.

If m is undefined, the line is vertical.

We have seen in the loaded beam experiment that we can derive the equation of a straight line from the graph by using the concept of slope and the coordinates of the point at which the line intersects the vertical axis. Now let us see how the slope and vertical intercept can help us to draw lines. Suppose a line has slope $\frac{2}{3}$ and a vertical intercept whose coordinates are $(0, 6)$. Let us draw the line as well as write its equation. To draw the graph, we start at the intercept $(0, 6)$. Then we use the slope to locate other points on the line. The fact that the slope is positive tells us that the line will rise as we go to the right, and the number $\frac{2}{3}$ tells us how "fast" the line rises. Between two certain points on the line, the vertical change will be two units "up" for a horizontal change of three to the "right". If we take the point which we know is on the line, $(0, 6)$, as one of the two points, we can find another point 3 units to the right and 2 units up. We can repeat this process as often as we wish, and quickly get several points through which we may draw the line. (Figure 20.)

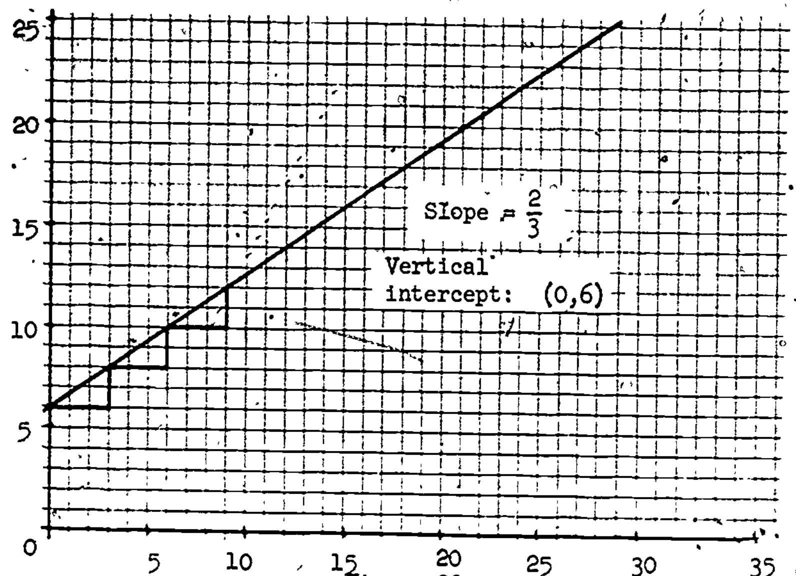
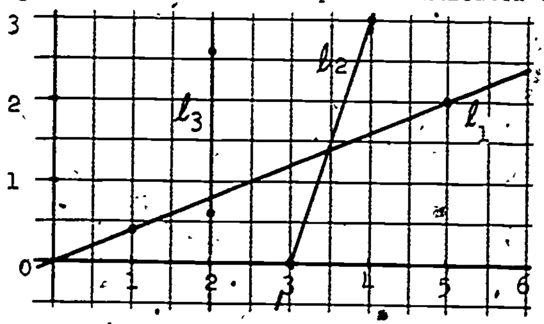


Figure 20

Since we now have $m = \frac{2}{3}$ and $b = 6$, we may write the equation of the line as $y = \frac{2}{3}x + 6$.

Exercise 4

1. Calculate the slopes of lines ℓ_1, ℓ_2, and ℓ_3 in the accompanying figure using in each case the two points indicated on the lines.



2. What is the slope of a horizontal axis? A vertical axis?

3. With reference to a set of coordinate axes, select the point (6, 3) and through this point

   (a) draw the line whose slope is $\frac{5}{6}$. What is an equation of this line?

   (b) draw the line through (6, 3) which has a slope of zero. What is an equation of this line?

4. Draw the following lines.

(a) a line through the point $(1,5)$ with slope $\frac{1}{2}$.

(b) a line through the point $(2,1)$ with slope $\frac{5}{2}$.

(c) a line through the point $(3,4)$ with slope 0.

(d) a line through the point $(4,3)$ with slope 2.

(e) a line through the point $(3,4)$ with slope undefined. (What type of line has no defined slope?)

5. Consider the line containing the points $(2,3)$ and $(9,5)$. Which of the following points is on this line? (Hint: First determine the slope of the line containing the points $(2,3)$ and $(9,5)$.)

(a) $(30,11)$
(b) $(7,4)$
(c) $(22,9)$
(d) $(23,9)$
(e) $(19,58)$
(f) $(58,19)$

6. Write an equation of each of the following lines.

(a) The slope is $\frac{2}{3}$ and the y-intercept number is 2. (The y-intercept number is the vertical coordinate of the point at which the line crosses the vertical axis. In this case, the coordinates of the intercept are $(0,2)$.)

(b) The slope is $\frac{3}{4}$ and the y-intercept number is 0.

(c) The slope is $\frac{2}{3}$ and the y-intercept number is $\frac{4}{3}$.

(d) The slope is 37 and the y-intercept number is 5.

7. What is the slope of the line containing the points $(0,0)$ and $(3,4)$? What is the y-intercept number? Write the equation of the line.

8. Verify that the slope of the line which contains the points $(0,5)$ and $(8,13)$ is 1. If (x,y) is a point on this same line, the slope could be written as

$$m = \frac{y - 5}{x - 0} \quad \text{or} \quad m = \frac{y - 13}{x - 8}.$$

Show that both expressions for the slope give the same equation for the line.

9. Write the equations of the lines through the following pairs of points. Use the method of Problem 8.

(a) $(0,3)$ and $(5,12)$
(b) $(5,8)$ and $(0,4)$
(c) $(0,2)$ and $(3,7)$
(d) $(5,8)$ and $(0,6)$
(e) $(3,0)$ and $(6,3)$
(f) $(3,3)$ and $(5,3)$
(g) $(3,3)$ and $(3,5)$
(h) $(4,2)$ and $(3,1)$

2.8 Relations and Functions

In the experiment which we performed, we collected a series of ordered pairs. In each of these ordered pairs, we noted a given load and a resulting position of the beam. We might have thought of a set of ordered pairs, (position of the beam, load), but in order to avoid confusion, we must always agree to state our ordered pairs in the same order. We consider load as the first element and position the second element of this set of ordered pairs.

However, a single ordered pair does not tell us very much. In fact, in order to get the complete picture, the mathematician and the scientist would prefer to have the total set of ordered pairs.

Any set of ordered pairs will be called a relation. The set of all first elements of the ordered pairs in any relation is called the domain of the relation. The set of second elements is called the range of the relation. We can say that a relation matches each element of its domain to one or more elements of its range. In the experiment with the loaded beam, the domain of the relation was the set of all possible loads while the range of the relation was the set of all possible positions of the pointer.

In Figure 21, we have a graph of a set of ordered pairs. This graph displays every ordered pair of the set and, hence, is a pictorial representation of a relation. If we check all the ordered pairs of this relation, we note that the first elements of these ordered pairs can be only real numbers from 0 through 10. This is the domain of the relation. The second elements of the ordered pairs are real numbers from 0 through 5, which is the range of the relation. The domain of the relation which is indicated along the horizontal axis is sometimes emphasized by the use of a heavy line, as in Figure 22. Range is indicated along the vertical axis and may also be emphasized with a heavy dark line.

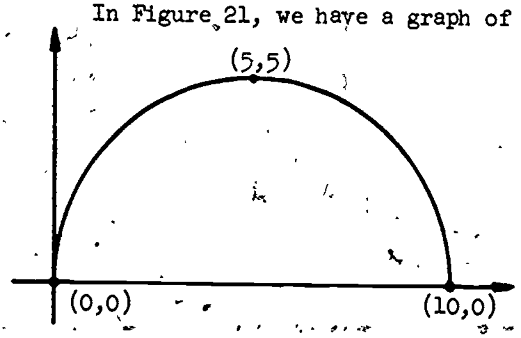


Figure 21

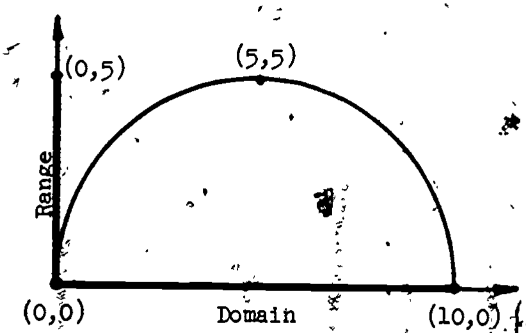


Figure 22

If we look at the relation in the experiment a little more critically, we will notice that it has some special properties. One of the properties leads us to predict that each time we load the beam in exactly the same way we will always expect to get exactly the same amount of bending. What does this mean in terms of our relation? Simply this, each load results in a single definite bending of the beam. Any time we have a relation which matches each element of the domain with exactly one element of the range, we give it a special name. We call this type of relation a <u>function</u>. Now then, to summarize what we have just said: a <u>function</u> is a set of ordered pairs such that each element of the domain appears in one and only one ordered pair.

Figure 13 gives a pictorial display of the ordered pairs which we predicted for this function. These were taken from experimental data. Again we emphasize the domain of the function which is indicated along the horizontal axis with a heavy line. In a similar manner, we can emphasize the range which is indicated along the vertical axis. (See Figure 23.)

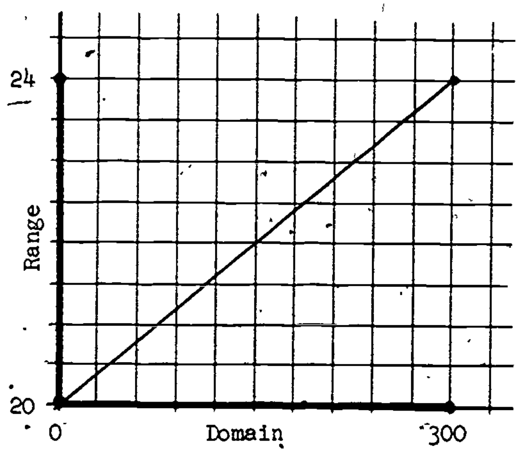


Figure 23

Exercise 5

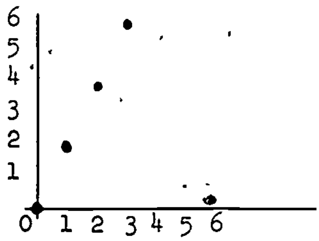
In Problems 1 through 5

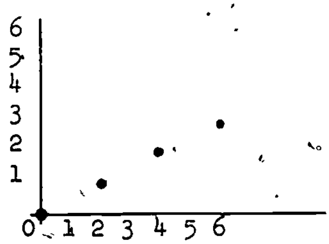
(a) Graph the ordered pairs given below, state the domain and the range and tell if the relation is a function.

(b) In each case form a new relation by interchanging the first and second elements of the ordered pairs. Graph this relation, state the domain and range and tell if it is a function.

Example: Given R = {(0, 0), (1, 2), (2, 4), (3, 6)}

New Relation: S = {(0, 0), (2, 1), (4, 2), (6, 3)}





domain {0, 1, 2, 3}
range {0, 2, 4, 6}
relation is a function

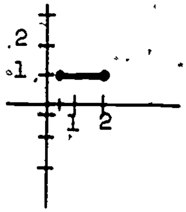
new domain {0, 2, 4, 6}
new range {0, 1, 2, 3}
new relation is a function

1. M = $\{(2, 3), (2, 4), (2, 5), (2, 6)\}$
2. N = $\{(0, 0), (1, 3), (5, 5), (9, 3), (10, 0)\}$
3. P = $\{(\frac{1}{4}, 4), (\frac{1}{2}, 2), (1, 1), (2, \frac{1}{2}), (4, \frac{1}{4})\}$
4. Q = $\{(5, 3), (8, 3), (11, 3), (14, 3), (17, 3)\}$
5. T = $\{(0, 2), (1, 1), (1, 3), (4, 0), (4, 4)\}$

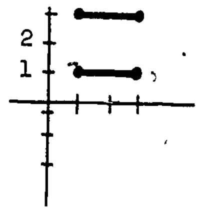
6. Which of the graphs of the relations shown below are graphs of a function?

Example:

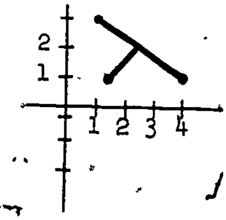
(1) <u>function</u>



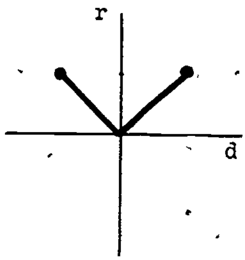
(2) <u>not</u> <u>a</u> <u>function</u>



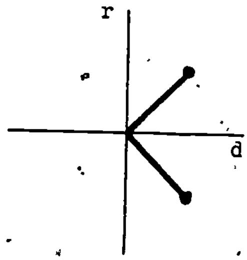
(3) <u>not</u> <u>a</u> <u>function</u>



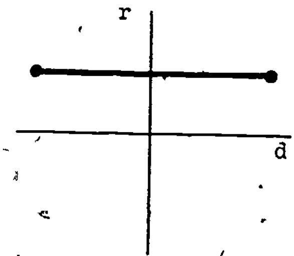
(a)



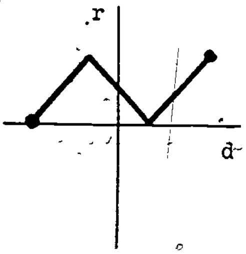
(b)



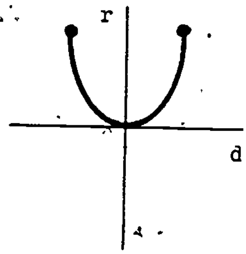
(c)



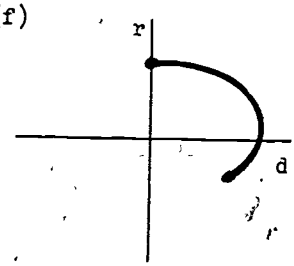
(d)



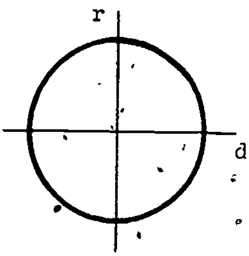
(e)



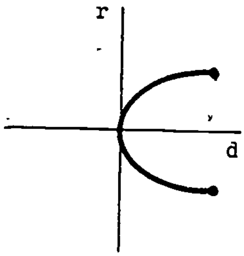
(f)



(g)



(h)



7. In the loaded beam experiment the data in the table forms a relation.
   (a) What are the domain and the range of this relation?
   (b) Is this relation a function?

8. Does the "best straight line" describe a function?

9. Are the domain and range of the "best straight line" relation the same as the domain and range of the "data relation"? Explain.

10. Are the domain and range of the equation found to represent the "best straight line" the same as the domain and range of the best straight line relation?

2.9 The Falling Sphere

This experiment continues our discussion of linear functions. We will encounter many of the concepts learned in the previous section. In addition, we will extend our knowledge of linear functions.

You may have learned in your study of science that all bodies take the same time to fall any given distance in a vacuum. You know, however, that an iron ball and a feather dropped at the same time from the same height will not reach the floor at the same time. Unless we drop objects in a vacuum, these objects always encounter some form of resistance from the medium through which the object falls. In a medium such as air or water this resistance is not constant, but increases with increasing speed. Eventually a point is reached when the upward resistive force equals the downward gravitational pull on the object. From this point on the object will fall at a constant speed. This speed is called the terminal velocity. A man jumping from a plane will reach a terminal velocity of about 120 miles per hour. A "sky diver" with proper control of his body can lower this figure to about 50 miles per hour. An opened parachute encounters a much greater resistance and lowers one's terminal velocity to a point of relative safety, about 20 miles per hour.

To investigate the phenomenon of terminal velocity, a small ball-bearing is allowed to fall through a thick fluid (Karo syrup). The ball-bearing will reach its terminal velocity in the first few millimeters and then the ball will continue to fall at a constant speed.

As in all experiments, we now have to think of all the possible conditions we are likely to meet, and decide how to handle them. Since our investigation will center around the speed at which the ball falls through the syrup, we must determine those conditions which influence this speed.

To test the influence of the size of the object upon the terminal velocity, we can drop ball bearings of different sizes into containers of the same size and shape, all filled with the same kind of liquid.

To test the effect of the jar upon the speed of the falling ball, we can drop the same ball in different size containers filled with the same type of liquid.

To test the influence of the liquid itself, we can drop ball-bearings of the same size into containers of the same size and shape but filled with different liquids.

If you notice any difference in the terminal velocity of the ball in any of these situations, then the factor that changed is a variable in which we are interested. Can you think of any other variables which may influence the experiment? Does the temperature of the liquid influence the speed of the ball in the same way that it affects the speed of the hot fudge moving off the top of an ice cream sundae?

Once we have our list of these conditions we must determine an experimental procedure in which we can control their influence on the terminal velocity. We will pick one container and one type of liquid and always have the ball fall in the same portion of the jar.

The terminal velocity of the ball, however, cannot be measured directly. What we must do is to measure the distance the ball will fall during some time interval. For example, to find the speed of an automobile, we have to know the distance traveled and the time taken to travel this distance.

In this experiment we will use a metronome as a timing device, thus providing an audible signal for selected time intervals. In this case we pick the time intervals, and the distances covered by the falling object will then depend on these time intervals (distance is a function of time).

To record the position of the ball as it falls through the syrup, fasten a thick paper tape to the side of the cylinder with cellophane tape. (See Figure 24.) Drop a ball-bearing into the cylinder so that it falls along the wall of the cylinder as close to one edge of the tape as possible. Since the velocity of the ball will be quite small, only a little practice is needed to follow the path of the ball along the edge of the tape and mark its position with a pencil each time you hear the click of the metronome. The metronome should be adjusted to click every second. Make a mark every other second.

A small magnet will be necessary to get the ball in position along the edge of the tape before releasing it. It is also used to bring the ball back to the surface for future trials. You do not have to mark the path of the ball for its entire fall. Ten position marks taken at two-second intervals will be sufficient for each trial.

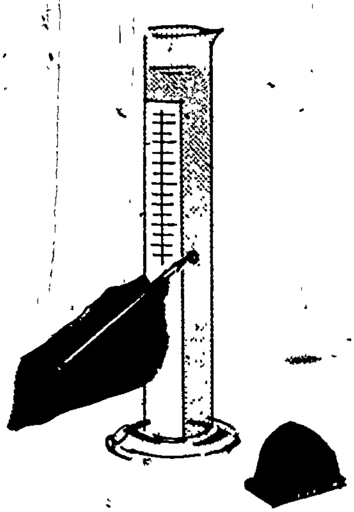


Figure 24

At least four separate trials of the experiment should be made using a new tape for each trial. Mark the trial number on the tape and indicate which end of the tape was at the top of the cylinder. It is not necessary to make the first mark in the same place each time. The first mark is taken to be the position of the ball at "zero" seconds, the second mark the position at the end of two seconds, etc.

2.10 The Graph and the Equation

After completing the four trials, fasten each tape in turn to a centimeter ruler so that the "zero" time mark coincides with one of the ruler marks. Measure the distance in millimeters from the "zero" mark to the first mark, from the "zero" mark to the second, etc. (See Figure 25.) Record the data for all four trials in tabular form and plot the resulting time-distance

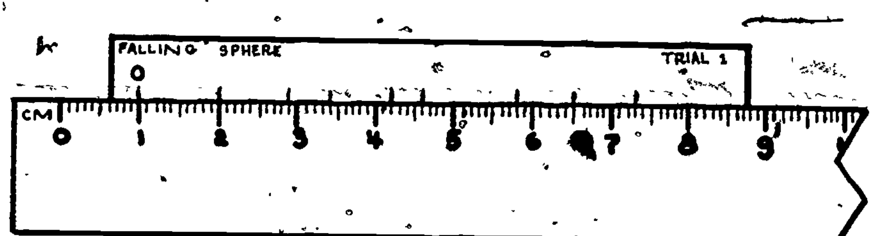


Figure 25

ordered pairs on a single sheet of coordinate paper. Since we have conducted the experiment in such a way that the distance traveled depends on the time interval, we will again follow conventional practice and label the horizontal axis "time in seconds" and the vertical axis "distance in millimeters". Remember to calculate the domain and the range before setting the scales on the paper. We again want the graph to "fill" the paper as much as possible.

If you make the "braid" arrangement discussed in the loaded beam experiment, all of the points should fall in some fairly narrow band (Figure 26). Do you think that if you were to repeat the experiment under the same conditions that your new points would fall within this band?

We obtain a band rather than a line because of the various errors in measurement and the influence of variables other than distance and time. The details of this analysis will be reserved for a future course.

There are many straight lines we could select to represent an idealized relationship between time of fall and distance.

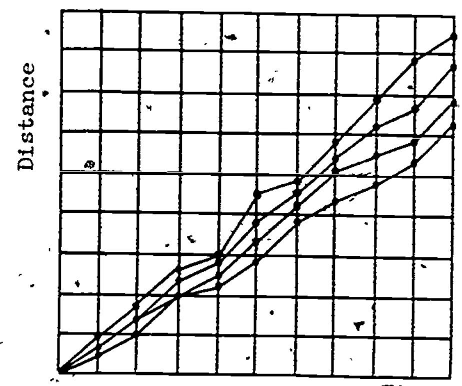


Figure 26

Draw what you consider to be the "best" straight line to represent the data. Remember to include the (0, 0) point in your line. The manner in which we performed the experiment tells us that at "zero" time the ball has fallen "zero" distance. Thus, even though there are many lines to choose from, every one of them should pass through the origin.

We still have to build a mathematical model of the physical relationship shown in our "distance versus time" graph. We can do this by repeating the procedure learned in the loaded beam experiment. The slope should not be difficult to compute at this stage. We know that the line must pass through the origin; hence the coordinates of the "y" intercept are (0, 0). The equation which describes the motion of the falling sphere is therefore quite simple. Calculate the slope using any two points on the line. Then, using the origin as the first point, and any arbitrary point on your line with coordinates (t, d) as your second, we have

$$\frac{d - 0}{t - 0} = m \quad \text{or} \quad \frac{d}{t} = m$$

and

$$d = mt.$$

The slope in this experiment has a special significance. In calculating the slope, the vertical distance from the first point to the second is a number of millimeters while the horizontal difference is a number of seconds. The slope therefore will be expressed in millimeters per second, and thus is a measure of the velocity of the ball. Since we have found that the experiment yields a straight line, the slope, and therefore the velocity, is a constant. Our initial comments are thus confirmed -- by the time we begin taking data the ball has already reached its terminal velocity and falls at a constant rate.

Exercise 6

1. Reproduce the "best straight line" you have drawn to represent the data of this experiment on a clean sheet of coordinate paper. Take the four pieces of paper tape used to mark the position of the ball and arrange them so that the zero marks are in line (Figure 27). On a clean fifth tape, make a mark to indicate a "zero" position and align this mark with

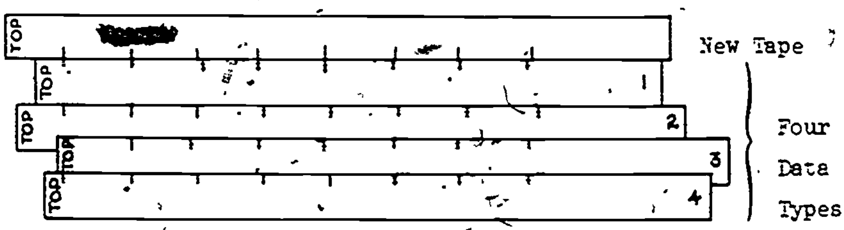


Figure 27

the other zero marks. The other marks on your tapes will not be "in line", but should tend to center in groups about a number of imaginary vertical lines. Make a mark on the clean tape to indicate your "guess" as to the position which best represents each vertical set of marks. Using the fifth tape as if it were a new trial, mark your measurements in the usual way, enter the data in your table, and graph the ordered pairs. Do these points come closer to forming a straight line than any of your four trial runs? How does this line compare with the "guess" you made from the "braid" arrangement?

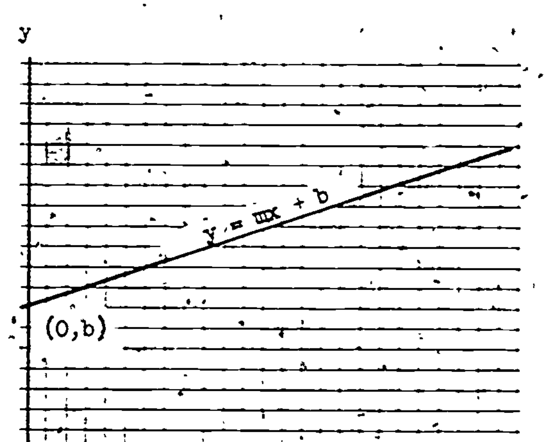
2. From the data of your four trials, find the average distance traveled by the ball in each time interval. Make a new column in your table, "Average Distance (mm)", and now plot average distance versus time on the same sheet of coordinate paper used for Exercise 1. How close do these points come to forming a straight line? You now have three lines on this sheet of coordinate paper. The first is the "best straight line" from your original data, the second is the line obtained in Exercise 1, and the third line is the one obtained by the process of averaging. How do these three lines compare?

3. Draw a graph using a scale of 1 second for each horizontal division and 1 millimeter for each vertical division. Draw a line which passes through the origin and has a slope of 1 mm/sec; 2 mm/sec; and 3 mm/sec. Label these lines.

4. Repeat the above exercise with a horizontal scale of 1 second per division but with a vertical scale of 0.5 millimeter per division. Are the two slopes the same?

2.11 The Point-Slope Form

When the data from the Loaded Beam Experiment was plotted on coordinate paper, a graph which resembled that in Figure 28(a) resulted, and we found that an equation of the form $y = mx + b$ could be used as a representation of this graph. In the Falling Sphere experiment, the graphical representation of the data passed through the origin (Figure 28(b)), and we found that all graphs of this type could be represented by an equation of the form $y = mx + 0$ since the graph passed through the origin. A simpler form of this equation is $y = mx$.



(a)

Suppose, however, we are to arrive at a graph which looked like that in Figure 28 (c). In this case, if our domain is limited to values greater than or equal to

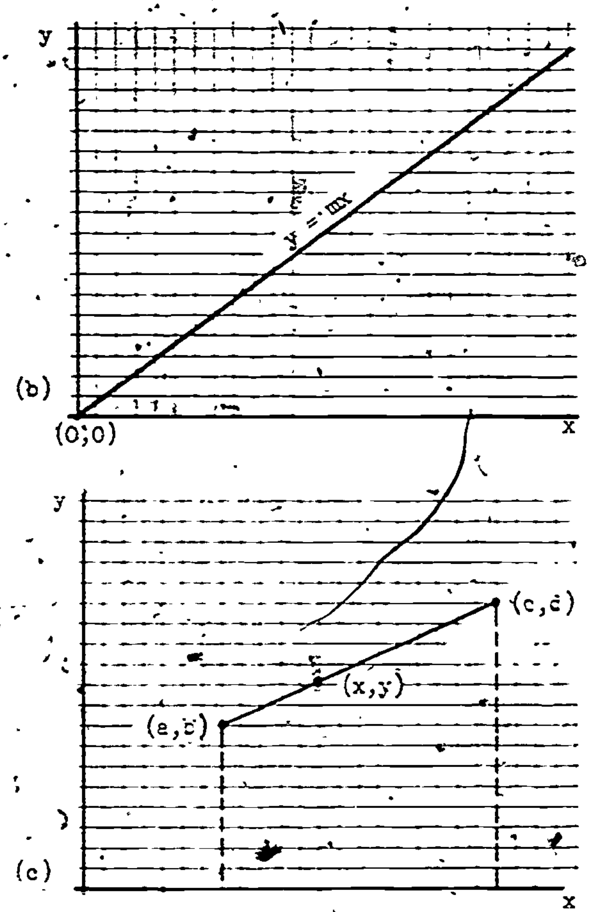


Figure 28

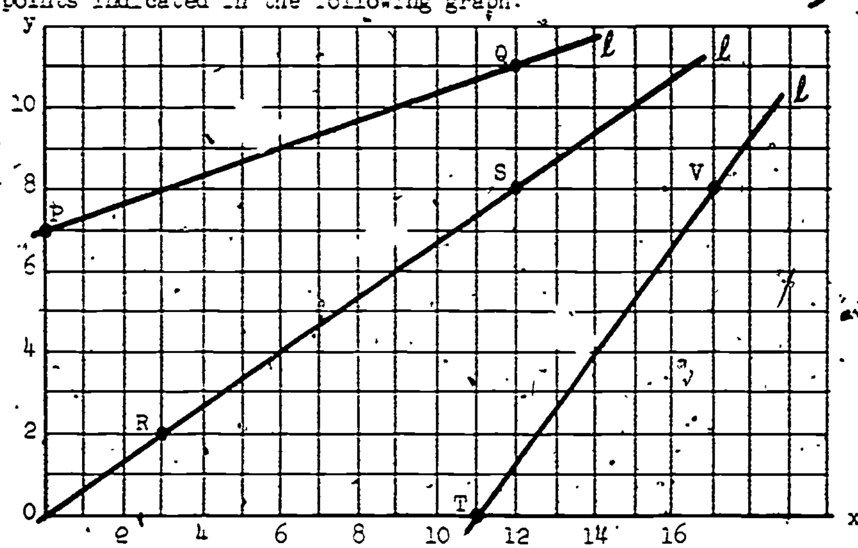
a, and less than or equal to c, we will not have a "y-intercept". The slope, however, can still be calculated in the usual way by selecting any two points on the graph and finding the ratio of the vertical distance between these points to the horizontal distance between them. The slope is the same for *any* two points on a straight line. To obtain the equation of this line, the end point of the segment which has the coordinates (a, b) is taken as our first point. Then for any arbitrary point with coordinates (x, y) we have

$$\frac{y - b}{x - a} = m.$$

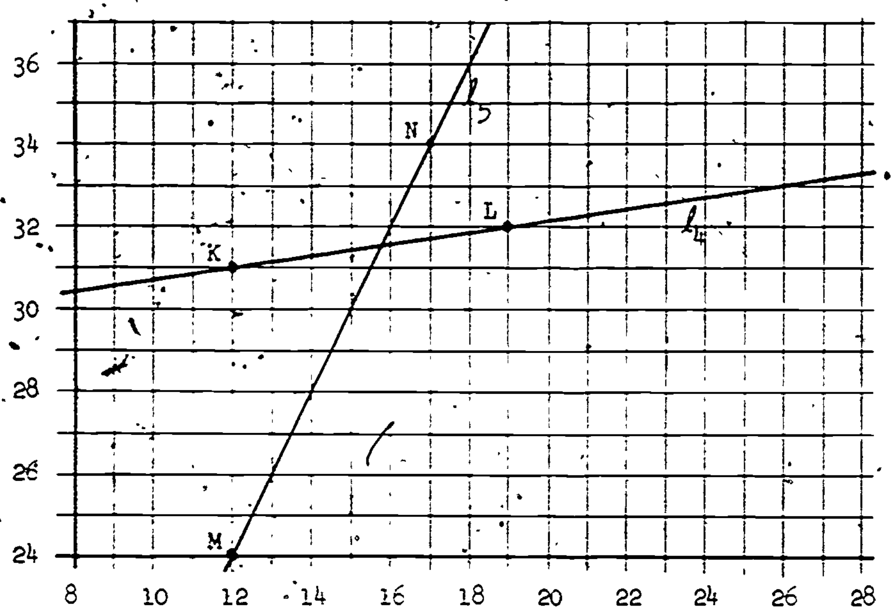
This is the third of three "special" forms of the equation of a straight line. This equation is known as the "point-slope" form of the equation of a straight line.

Exercise 7

1. Write the equations of the lines ℓ_1, ℓ_2, and ℓ_3 using the two points indicated in the following graph.



2. Write the equation of the lines ℓ_4 and ℓ_5 using the points indicated in the following graph.



3. Refer to your load-position graph obtained in the loaded beam experiment. Using a point not on the vertical axis together with the slope, find the equation to represent the best straight line. Show that this is equivalent to the equation obtained, using the slope-intercept form.

4. State the slope of the graph of each of the following equations. Give the coordinates of three points on the graph of each.

(a) $\frac{y - 6}{x + 4} = 3$

(b) $\frac{y - 2}{x - 3} = \frac{3}{5}$

(c) $\frac{y - 10}{x - 10} = 1$

(d) $2\left(\frac{y - 5}{x - 15}\right) = \frac{2}{3}$

(e) $5y = 2x - 15$

(f) $3x + 5 = y$

(g) $y + 4 = \frac{5}{2}(x - 2)$

(h) $\frac{x - 2}{y - 5} = \frac{9}{10}$

2.12 Summary

In this chapter, we developed the concept of drawing the graph of the "best line" from experimental data. This "best line" graph was called the physical model of the experiment. A mathematical equation, called the mathematical model, was then derived from the physical model.

In this particular chapter linear physical models were discussed. The slope of a straight line was defined as $\frac{\text{rise}}{\text{run}}$.

Three forms of a linear equation were developed:

(1) Equation of a line passing through the origin,

$$y = mx.$$

(2) The slope-intercept form of the equation of a straight line,

$$y = mx + b.$$

(3) The point-slope form of the equation of a straight line,

$$\frac{y - b}{x - a} = m \text{ for } x \neq a,$$

A relation was defined as a set of ordered pairs. The set of all first elements of these ordered pairs is the domain of the relation and the set of all second elements is the range of the relation.

A set of ordered pairs, where each element in the domain appears in exactly one ordered pair, defines a function.

Chapter 3

TRAMPOLINES AND GASES

3.1 Introduction

Many times, even though the data in an experiment is nonlinear in character, it is possible to compute new data which is of a linear type. The following experiment on the trampoline is an example of this type of experiment.

At one time or another you may have had the opportunity to jump on a trampoline. If so, you know what fun it can be. The "springiness" of the trampoline permits you to execute flips and turns not possible under other circumstances. The question now is: Do you suppose it would be possible to make a mathematical analysis of your behavior on a trampoline?

As with many other physical situations, this one seems much too difficult to handle. A person on a trampoline not only bounces from the canvas, he usually jumps at the same time to give his body extra height. He also twists his body and swings his arms in a way that will produce the maneuvers he desires. All this is extremely complex behavior.

If we are to learn anything at all about a trampoline, we must somehow simplify matters considerably. Perhaps we could eventually learn about one's entire behavior on a trampoline through a series of experiments, each one designed to examine one aspect, and one aspect only, of the entire situation.

3.2 The Trampoline Experiment

For our purpose a $\frac{5}{8}$-inch glass ball (marble) will be dropped on a miniature home-made trampoline. We will carefully examine the way in which the ball bounces. There will be no flexing of one's legs or flailing of arms, just a simple bounce, bounce, bounce, ... on the trampoline.

A suitable trampoline is made by stretching a ten- or fifteen-cent balloon with its neck cut off over a 9- or 10-inch pie plate. This forms a highly-stretched elastic membrane that serves beautifully as a trampoline for the glass ball.

After dropping the ball a few times on the trampoline, from a height of about 50 cm, the basic behavior of the ball is clearly seen. The ball will bounce 25 to 30 times before finally coming to rest upon the rubber membrane. With each bounce the ball will rise to a maximum height that is somewhat less than the height to which it bounced previously. If this motion were to be "stretched out" sideways on a flat surface, it would appear as shown in Figure 1.

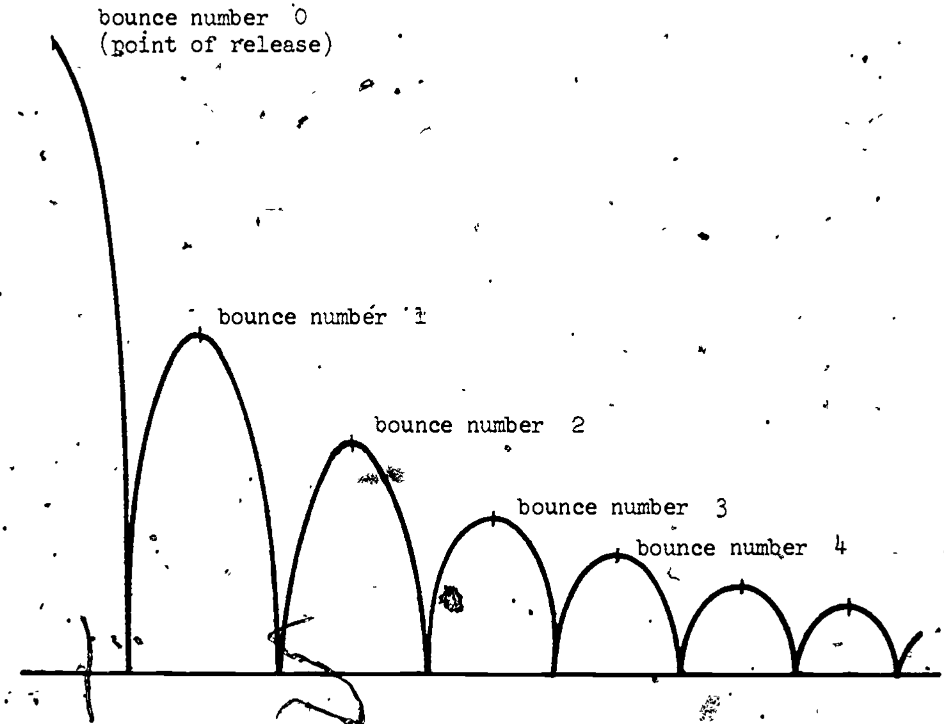


Figure 1. Path of a bouncing ball

Even with the trampoline situation simplified to the point of using a ball in place of a human being, the bouncing of the ball is still a rather complicated affair. The ball picks up speed as it descends, makes a small dimple in the trampoline as it hits, and then flies upward with ever-diminishing speed until it reaches the top of its bounce and begins the sequence all over again. Affairs can be simplified still further by fixing our attention only upon the maximum height to which the ball ascends with each bounce. We choose to ignore the condition of the ball at all other times.

We have selected for study a limited part of the entire behavior of the bouncing ball. To what maximum height does the ball rise with each bounce? As before, we will attempt to list all the variables which might conceivably influence these heights, and then permit one and only one of these variables to change during the experiment. Certainly the height from which the ball is dropped will influence the height to which the ball rebounds. The mass and size of the ball itself may also influence the situation. And then of course the nature of the rubber membrane and how tightly it is stretched will also influence the maximum heights to which the ball bounces. Can you think of any other influences?

All of the variables mentioned can be kept constant as the ball bounces -- the height from which the ball is dropped, the size and mass of the ball and the condition of the trampoline itself. And yet under these conditions the height to which the ball rebounds with each bounce still changes. What variable, then, influences this height? In case you've missed it up to now, it is the number of bounces the ball has taken. In other words, the maximum height to which the ball rises with each bounce most certainly depends upon the number of bounces the ball has made.

The experiment can now be designed in a way that will permit us to make fairly accurate readings of the maximum height of the ball with each bounce. This height can be read more easily and accurately by using a shadow of the ball rather than the ball itself. A 150- or 200-watt bare bulb should be placed four to six meters from the trampoline so that the shadow of the ball will be cast in a nearly horizontal direction. The bulb should be placed at a height which is close to the middle of the bounce heights that will be recorded.

The shadow of the ball will be projected on a white card upon which a centimeter scale is drawn. The card should be placed directly behind the trampoline. The card should be as wide as the pie plate and at least 50 cm in height. The rulings should be drawn carefully across the entire card for each centimeter of height. Every fifth line should be drawn darker for easy reading, and marked. With this arrangement one should be able to read the position of the top of the ball's shadow to 1 mm (one-tenth of a division). The entire experimental arrangement is shown in Figure 2.

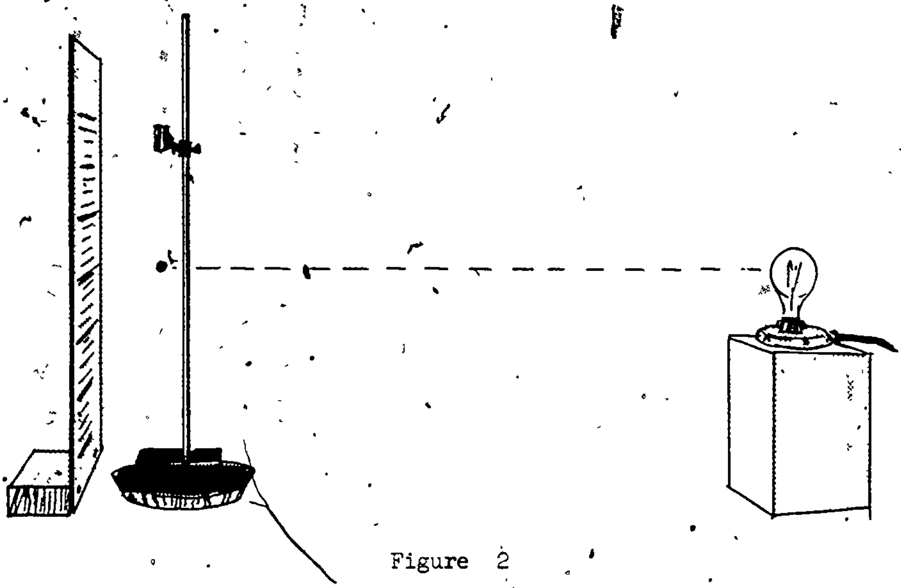


Figure 2

The pie plate must be accurately level or the ball will bounce off the trampoline. Three small pieces of modeling clay placed beneath the pie plate will make this adjustment easier. Place the ball on the trampoline and adjust the pie plate until the ball will not roll off in any direction. (A small bubble level could also be used.) A ring stand equipped with a burette clamp is used for releasing the ball from a height of about 50 cm. Be sure the clamp is rubber-covered, and then tighten it until it will just hold the ball. A slight push will now send it on its way. The position of the clamp over the trampoline must be adjusted so that the ball will continue to bounce from the trampoline for at least ten bounces. Some final leveling of the pie plate may also be needed.

Now we are ready to record data from the experiment. Label the first column on your data sheet "bounce number (n)" and the second column "height in cm (h), trial 1". The first recorded bounce number will be number 0. The corresponding height will be the height of the ball at the point from which it is released. The initial position of the ball is found by observing the shadow of the ball when it is still in the clamp. Be sure that the ball is in the position it occupies just before it slips out of the clamp. If only the bottom of the ball casts a shadow, observe this height and add the ball's diameter.

Corresponding to bounce number 1 will be the maximum height of the ball after the first bounce. Make four observations of the height of the first bounce before continuing to the height readings of the second bounce. Record these four trials in columns 2 through 5. Starting with the ball at the same point of release, now let the ball bounce twice and make four observations of the second bounce height. Discard data obtained when the ball obviously takes a bad bounce. Also, do not begin to record data until the approximate height of rebound is known. In this way you will accumulate four readings for each of 10 bounce numbers (see Figure 3).

Average your four height readings for each of the 10 bounce numbers and place each average in column 6 of your data sheet. If your centimeter scale placed behind the trampoline rested upon the desk, you must now subtract the height of the trampoline above the desk from each of these averages (and from the release height) to obtain heights above the trampoline membrane. Place these "corrected heights" in column 7.

The data function obtained now consists of the ordered pairs displayed in columns 1 and 7. The first elements of these pairs are the bounce numbers while the second elements are the maximum heights above the trampoline. Before attempting to analyze the situation further, display these points on a sheet of coordinate paper plotting bounce number (n) along the horizontal axis and height (h) along the vertical axis. The graph obtained should appear similar to the graph shown in Figure 4.

THE TRAMPOLINE

Bounce number (1) n	Height in cm (h) (2) trial 1	(3) trial 2	(4) trial 3	(5) trial 4	Average height (6) $\bar{h}$	Corrected height (7) h_n	(8) h_{n+1}
0		(obtained from shadow)					
1							
2							
3							
4							
5							
6							
7							
8							
9							
10							

$\frac{5}{8}$-inch glass ball (marble) (diameter 1.4 cm)

light source = 4 meters distant

Figure 3

THE TRAMPOLINE

Height in cm (h)			Average height	Corrected height		Calculated heights
(3) trial 2	(4) trial 3	(5) trial 4	(6) h	(7) h_n	(8) h_{n+1}	(9) h_{n+1}
(obtained from shadow)						

(marble) (diameter 1.4 cm)

eters distant

Figure 3

3.3 Function of Integers

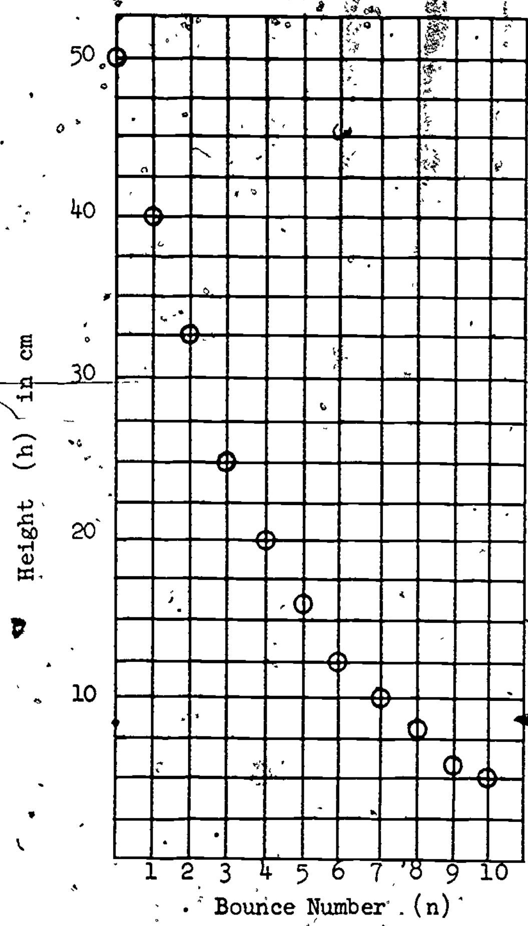


Figure 4

The function displayed in Figure 4 is a physical model of the experimental situation, for it is nothing more than a graph of the data function. It remains to find a suitable mathematical model to represent the trampoline behavior.

Did you draw a "best curve" through or near the points? This may have become a habit arising from past experience.

In many experimental situations, the drawing of a "best line" or curve is completely justified. In these cases we could assume that values in the domain of the function could have been selected which would yield corresponding intermediate values in the range of the function.

The values in the domain of our trampoline function, however, are the so-called "bounce numbers". Can we have a bounce number 2.6 for instance, and will there then be a corresponding maximum height to which the ball bounces? Think about this question for a moment, and refer back to Figure 1 where the general behavior of a bouncing ball is indicated.

Will you not agree that the peak heights to which the ball bounces from the trampoline correspond only to the integers, and not to intermediate numerical values? The domain of our function includes only the integers 0, 1, 2, 3, ... , but the range of the function includes positive real numbers which are not necessarily integers. We choose to call this kind of relationship a function of integers.

If you fell into the trap and drew a "best line" or curve through the points, that line must now be erased, for it has no significance.

There are many examples of functions of integers both in everyday experience and in formalized physical science. In most cases, the integers are thought of as belonging in the domain of the function, but in certain cases the range of the function may also be integers. The times of sunrise, sunset, moonrise, and moonset for a given locale are functions of integers (the day of the year). The thickness of a book depends upon the number of pages. The height of a building may be expressed as a function of the number of stories.

Exercise 1

1. Can your graph of bounce number (n) and bounce height (h) be used to interpolate values of the height for non-integral values of the bounce number? Explain.
2. Why can your (n,h) relation be referred to as a function?
3. Do you think that the (n,h) graph can be extended to find values of the maximum bounce heights for bounce numbers greater than 10? If so, to what value of n would you be willing to go?
4. Construct a graph that shows roughly the time of sunset for each day of this week.

3.4 Mathematical Trampoline Model

The trampoline function is a function defined for bounce numbers (or integers) and presents a mathematical situation which differs from previous situations. For this reason, the procedures we have used before may be of little help to us in this situation. We need a new procedure.

As is so often the case, the hints we need to develop the mathematics of a particular experimental situation can be found from an analysis of the experiment itself. In this case we need to re-examine the bouncing from the trampoline to give us clues as to the manner in which the mathematics might develop.

The ball was released from a fixed height, bounced once, rose to a maximum height, bounced again, rebounded to another maximum height, and so on. We might well ask the following question: What is the physical distinction between the first bounce and the second, or between the second bounce and the third? After the ball bounced once, for example, we could have caught the ball at its maximum height and later released it at this same height. This would, in a sense, start the experiment all over again, the only difference being that this time the ball would have been released at a lower height.

There must, then, be some relation between the height to which the ball bounces and the height to which it bounced the time before. Suppose that the point of release, the size and mass of the ball, and the properties of the trampoline are all held fixed. It then seems reasonable to assume further that the height to which the ball bounces depends only upon the previous bounce height and nothing else. The two preceding statements are extremely important ones. In effect, they constitute a hypothesis about the physical behavior of the bouncing ball.

Our search for a mathematical description of the bouncing ball can now be concentrated in a single direction. We seek a relation between successive bounce heights.

Before pursuing the subject further, it will be helpful to introduce some new mathematical notation. Call the height to which the ball rises after bounce number n the height h_n. The small subscript is the bounce number corresponding to the height, and serves as a reminder as to which of the maximum heights we refer. For example, h_0 is the height for the zeroth bounce (the point of release), h_1 is the height corresponding to the first bounce, h_2 is the height corresponding to the second bounce, and so on.

The relation we wish to find can now be restated using this notation. We seek the relation between h_{n+1} (any maximum bounce height corresponding to the bounce number $n+1$) and h_n (the height of the previous bounce). In other words, the height of the 8th bounce, h_8, depends upon the height of the 7th bounce, h_7; likewise the height of the 3rd bounce, h_3, depends upon the height of the 2nd bounce, h_2. For these relations, $n = 0, 1, 2, 3, \ldots, 9$.

Let us summarize the state of affairs at the moment. We have already obtained experimentally a relation between h_n (the values in column 7 of Figure 3) and n (the values in column 1 of Figure 3), and this relation was displayed on coordinate paper (Figure 4). We found it to be a "function of integers". Now, however, we wish to find a new set of ordered pairs. The first element is h_n and the second element is h_{n+1}. For example, if $n = 6$, then $n+1 = 7$ and h_6 is the height of the 6th bounce and h_7 is the height of the 7th bounce. The ordered pair (h_n, h_{n+1}) would be (h_6, h_7) in this example.

The relations of Figure 4 and the ordered pairs (h_n, h_{n+1}) are different, but both are part of the same over-all problem. The second relation is the one indicated by the ordered pairs shown below,

$$(h_0,h_1)$$
$$(h_1,h_2)$$
$$(h_2,h_3)$$
$$(h_3,h_4)$$
$$(h_4,h_5)$$
$$(h_5,h_6)$$
$$(h_6,h_7)$$
$$(h_7,h_8)$$
$$(h_8,h_9)$$
$$(h_9,h_{10})$$

since we already have all the first elements in the above relation tabulated in column 7. All we need to do now is to tabulate the second element in column 8 of Figure 3. Once this is done, plot these ten points on a clean sheet of coordinate paper. Plot the values of h_n along the horizontal axis, and the values of h_{n+1} along the vertical axis.

The graph may surprise you. Don't you find that these points, allowing for some experimental errors, are arrayed fairly well along a line? A typical result is shown in Figure 5.

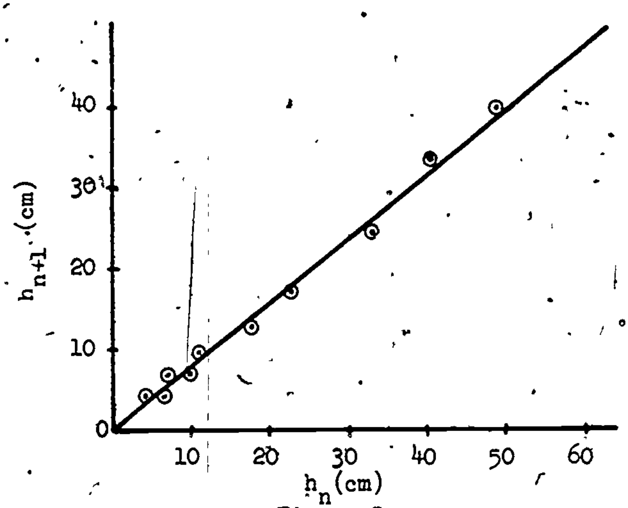


Figure 5

Did you draw a "best straight line" through the points? Perhaps not, for now you may be a little suspicious of such a procedure. By drawing in the line we may be suggesting that there are maximum bounce heights corresponding to any positive number, not just to the integers.

It is true that in our experiment there were eleven h_n's and only eleven, with no heights in between. Looked at more broadly, however, the relation shown in Figure 5 is a relation between any bounce height and the one preceding it. The height of the preceding bounce could have had any value, a value that would depend upon the height of the release point for the ball and the characteristics of the trampoline surface. Thus the elements in the domain of the relation (the h_n values) could assume any value, and corresponding to this value there would be a corresponding element in the range of the relation (an h_{n-1}).

"Filling in the line", therefore, is a procedure that is justified in this situation. In case you did not draw this line before, draw it now. Make sure that your best line passes through the origin, for most certainly a bounce height of zero will produce a zero height on the next bounce.

From our best line we obtain the equation

$$h_{n+1} = m \cdot h_n$$

where m is the measured slope of the line. There is an important physical interpretation of the slope of this line. If we solve for the slope to obtain $m = \frac{h_{n+1}}{h_n}$, we see that the slope is the ratio of any maximum bounce height to the height of the previous bounce. The value of this ratio (slope) will determine how quickly the successive bounces of the ball from the trampoline will "die away".

We must remember that in the above equation, the value h_0 has been determined by the experimental arrangement. h_0, together with m, are the two constants we need to calculate any of the bounce heights corresponding to bounce numbers 1, 2, 3, ..., 10 is indicated below.

$$h_1 = mh_0$$

$$h_2 = mh_1$$

$$h_3 = mh_2$$

$$\vdots$$

$$h_9 = mh_8$$

$$h_{10} = mh_9$$

Each right-hand of an equation is obtained from the right member of the preceding equation by multiplying by m. The process can be repeated, therefore, to give h_{10}:

$$h_{10} = mh_9 = m(mh_8) = m^2(mh_7) = m^3(mh_6) = m^4(mh_5) = m^5(mh_4) = m^6(mh_3)$$
$$= m^7(mh_2) = m^8(mh_1) = m^9(mh_0)$$

that is, $$h_{10} = m^{10}h_0.$$

Similarly, we can find all of the ten bounce heights once we know the two constants m and h_0. It is easy to see that a single equation can be written to obtain h_{n+1} for any n:

$$h_{n+1} = m^{n+1}h_0 \qquad \text{for} \qquad n = 0, 1, 2, \ldots, 9 .$$

To test whether this equation is, in fact, as good as the previous ten equations, we have only to set n = 9, for example, and find that $h_{10} = m^{10}h_0$. For n = 3 we would have $h_4 = m^4h_0$. Since n has ten values, we have ten equations.

The procedure we have used has turned out in a most interesting way. We started with the equation $h_{n+1} = m \cdot h_n$. This equation represents a linear relation between h_n and h_{n+1}. The equation we have now obtained is quite different, for it contains but one h-value. It is an equation which expresses the relation between n and h_{n+1}. It is certainly not a linear equation. It is the relation that we are seeking from the very beginning. We have already graphed the experimental relation between n and h_{n+1} (Figure 4) and now we at long last have a mathematical model of this relation.

Use the equation $h_{n+1} = m^{n+1}h_0$ to calculate new values of the heights corresponding to each bounce number and place these in column 9 of Figure 3. Now plot these on your graph (similar to Figure 4) and compare the values predicted in this way by our equation with the values obtained experimentally. The two sets of points should agree rather well.

Let us assume $h_3 = 27.7$ cm and $h_4 = 23.0$ cm, then $m = \frac{h_4}{h_3} = \frac{23.0 \text{ cm}}{27.7 \text{ cm}} \doteq 0.830$ where $\doteq$ means approximately equal. As a result we may take $m \doteq 0.8$ as a value for computation. Since $h_0 = 50.0$ cm

$$h_{10} \doteq (0.8)^{10}(50.0 \text{ cm})$$

$$h_{10} \doteq (0.1)(50.0 \text{ cm}) \quad \text{where } (0.8)^{10} \doteq 0.1$$

therefore $$h_{10} \doteq 5.0 \text{ cm} \quad \text{(calculated value)}$$

Only one consideration remains. In the experiment only 10 bounces were observed. We found a mathematical expression that accurately describes the bounce heights that were obtained. Will our equation continue to describe the heights to which the ball ascends after 100 bounces, 1000 bounces, or even more? One does not have to look very far to find the answer. The ball will not continue to bounce indefinitely. Our equation must at some point cease to describe the situation. Physically we can lay blame on the ever-present friction between the ball and the trampoline. The frictional forces present bring the bouncing to a stop.

We see that the domain of bounce heights cannot be extended indefinitely. The domain includes 10 bounces and no more. A new experiment would have to be performed to determine whether our equation properly predicts the behavior of the ball for a greater number of bounces.

3.5 Experimental Extension

Now that an analysis of the trampoline function has been made, we must remember that the entire problem utilized the data obtained with the glass ball. We have not faced the question as to whether a different type of ball would give different results. It is most interesting to replace the glass ball with a nylon bearing of about the same size and repeat the experiment.

Using the same experimental arrangement and procedure that was used before, adjust the level of the trampoline so that the nylon bearing will continue to bounce from it for at least four or five times when released from a height of about 50 cm. Record the data just as you did before, but on a new data sheet. When you graph the relation between h_n and h_{n+1} this time, however, you will find the slope to be somewhat lower than it was for the glass ball.

The slope that is obtained is somehow a characteristic of the ball that is used, for one value is obtained for the nylon ball and another for the glass ball. The difference in behavior for the two balls is immediately evident from the way they bounce on the trampoline. The glass ball will continue to bounce a very great number of times (if it doesn't jump off the trampoline) compared to the number of bounces for the nylon bearing.

Be sure to plot the (n,h) data pairs for the glass ball on the same coordinate paper used to represent the data for the nylon ball. Can you now anticipate about where the points would fall on this graph when $h_0 = 50$ cm and $m = 0.4$, $m = 0.6$, or even $m = 0.9$?

Exercise 2

1. Referring to your graph of the (h_n, h_{n+1}) relation, what is the domain and range of the experimental data? What restrictions if any would you place upon the domain and range of the mathematical equation found to represent the line?

2. Suppose that

$$h_{n+1} = (0.5)^{n+1} h_0 .$$

Sketch to the same scale a series of (n,h) points for $h_0 = 10, 50$ and 100.

3. Suppose that

$$h_{n+1} = m^{n+1} \cdot 100 .$$

Sketch to the same scale a series of (n,h) points for $m = 0.3$, 0.6 and 0.9 .

4. Make a possible interpretation of the significance of the equation

$$h_{n+1} = (0.5)^{n+1} \cdot 100$$

for the case $n = -1$.

5. Why did the domain of the relation

$$h_{n+1} = m^{n+1} h_0$$

include the value $n = 9$ and not $n = 10$?

6. What is the physical unit of the quantity "m" in the equation

$$h_{n+1} = m^{n+1} h_0 \ ?$$

7. Do you think it would be possible to find a value of m greater than or equal to 1 ? Explain.

3.6 Gay-Lussac's Law

Scientists are often prone to state their discoveries in terms of "laws". Mathematicians, on the other hand, discover "theorems". The present experiment has to do with gases and gas pressures. The physical law that is involved is not important for our purposes, but the mathematics that stems from an analysis of the experiment is.

The apparatus that will be used to investigate gas pressure is an extremely simple one. The equipment is shown in Figure 6. It consists of a copper bulb connected through a small pipe to a pressure guage at the top. The system was sealed off at a time when it contained ordinary air at atmospheric pressure. The guage is numbered to read pressure in pounds per square inch. The pressure reading corresponds to the pressure of the air within the bulb, nothing else, for this air is completely sealed off from the outside air.

Whether you already know about gas pressure or not does not matter. In this experiment it will simply be a number read from the guage.

Figure 6

If the pressure within the gas enclosed by our apparatus is to be measured, we must find some way to influence that pressure before we can learn something of significance. Perhaps we can squeeze the copper chamber. This external pressure might cause the pressure within the gas to rise. The metal bulb, however, is a fairly rigid container and we would have to damage the bulb before we could get a measurable pressure change indicated on the guage. If we were to change the temperature of the gas within the copper bulb, on the other hand, the change in temperature of the gas might very well change the pressure within the gas. Since copper is a good conductor of heat, the apparatus seems ideally suited for conducting heat either into or out of the gas. All we need to do is to immerse the bulb in water. Whatever the temperature of the water, the temperature of the gas inside will soon be the same.

This, then, is the design for an experiment. For each temperature of the gas we will read a corresponding pressure. We may find as many ordered pairs as we wish and the set we collect will then be a function. It is convenient to collect data at about ten-degree intervals between $0^{\circ}C$ and $100^{\circ}C$. A thermometer placed in the water surrounding the bulb measures these temperatures. The "C" stands for "Centigrade" readings, and the two extreme temperatures correspond to the freezing and boiling temperatures, respectively, of water. This procedure gives us ten or eleven ordered pairs (C,P), that is, Centigrade temperature - pressure pairs.

It is important to note that in this experiment, the gas is influenced only by the temperature, nothing else. For example the volume of the gas is held constant throughout. It would be difficult to change the volume even if we wished to do so. Can you think of other possible influences upon the pressure of the gas?

Record the Centigrade temperatures in the first column of your data sheet and the corresponding pressure readings in the second. Be sure to label the pressure column with "pounds per square inch", for this is the unit of pressure read from the gauge. If you help conduct the experiment, be sure to estimate a reading to the nearest tenth of the smallest division, both on the thermometer and on the pressure gauge. If the space between the smallest divisions on the pressure gauge represents 2 pounds per square inch, a tenth of this division will then represent 0.2 pounds per square inch. This tenth's rule is a good "rule-of-thumb" to follow. With practice you will be able to make readings to the "nearest tenth" in most cases. Be suspicious, however, of a person who claims to be able to read more closely than this.

As always, we will want to graph our function before we attempt to analyze it further. Since the temperature readings are elements in the domain of the function, and the corresponding collection of pressure readings constitute the range, plot Centigrade temperature along the horizontal scale and pressure in pounds per square inch along the vertical scale. Select temperature and pressure scales that will make the graph as large as possible. Allowing for some inaccuracies in the data, we see that the points lie on or near a straight line. When you draw this line, you are assuming that we might have selected any temperature intermediate between those actually taken, and that these would then determine corresponding pressure readings.

Although the same data is used by everyone, the line that you draw to represent the experimental function will most certainly not be the same line as that drawn by someone else. This is as it should be, for your judgment as to the "best" line will differ from his.

The graph of the relation between temperature and pressure obtained will look something like the graph shown in Figure 7.

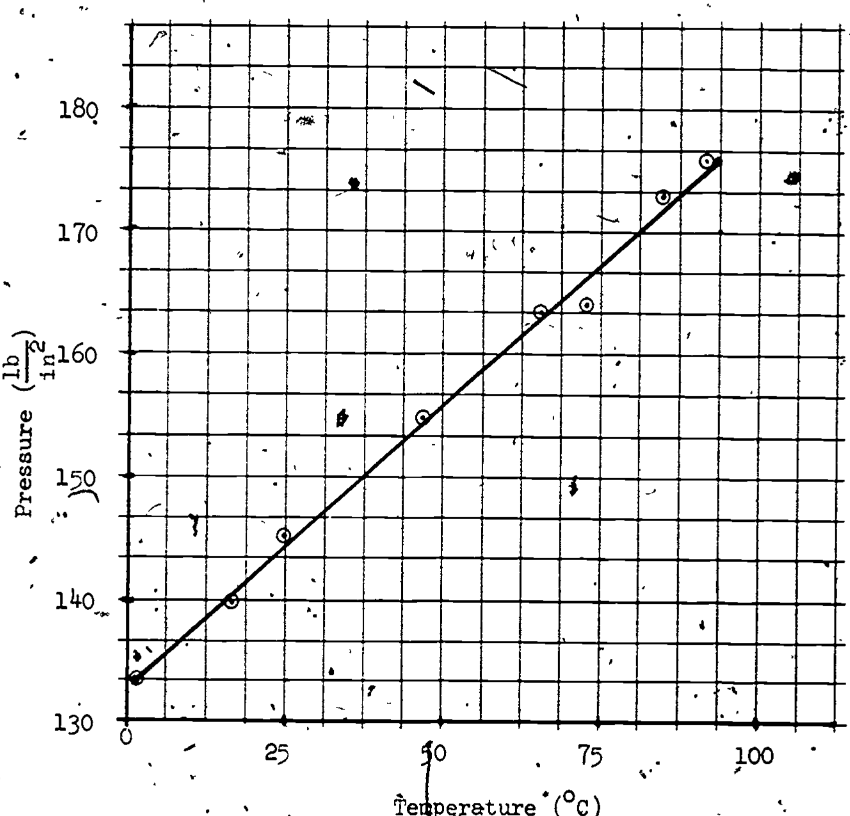


Figure 7

The relationship shown has a special name -- Gay-Lussac's Law. In words it can be stated as follows: For a gas held at constant volume, the pressure of the gas varies linearly with the temperature.

Mathematically, we have learned to express this linear relationship in the following way:

$$P - d = m(C - c) .$$

In this expression, c, d, and m are constants that we can determine from the graph.

3.7 Extending the Temperature Domain

The domain of our linear function has been set by the conditions of the experiment. The domain we have used is the set of all temperatures in the interval $0 < C < 100$. When we predict a pressure corresponding to an arbitrarily selected temperature within this interval, the process is one of interpolation. If we attempt to predict a pressure that corresponds to a temperature outside this interval at either end, the process is one of extrapolation.

This experiment provides us with a golden opportunity for extrapolation. Notice that within the domain $0 < C < 100$, the pressure diminishes linearly with a drop in the temperature. Do you suppose it would be possible to reduce the temperature by such a large value that the pressure would actually drop to zero? Although you have no way of knowing the proper response to this question, we can find the temperature at which the pressure would fall to zero IF the gas continues to behave in the same manner in the temperature region below $0^{\circ}C$. The "IF" here is a very big one.

To extrapolate graphically to lower temperatures (that is, to extend the domain of the function), make a new plot on a fresh sheet of coordinate paper. The temperature scale must span an interval of about $400^{\circ}C$, from a negative $300^{\circ}C$ on the left to a positive $100^{\circ}C$ on the right. The vertical scale of pressure must now extend upwards from a pressure of 0 to the maximum pressure previously obtained. Graph your original ordered pairs on this new sheet. Now draw a line "through" these points downward and to the left until it intersects the horizontal axis. The temperature value of this point of intersection (pressure = 0) will represent the temperature of the gas that would, presumably, reduce the gas pressure to zero. This new graph should look something like the one shown in Figure 8.

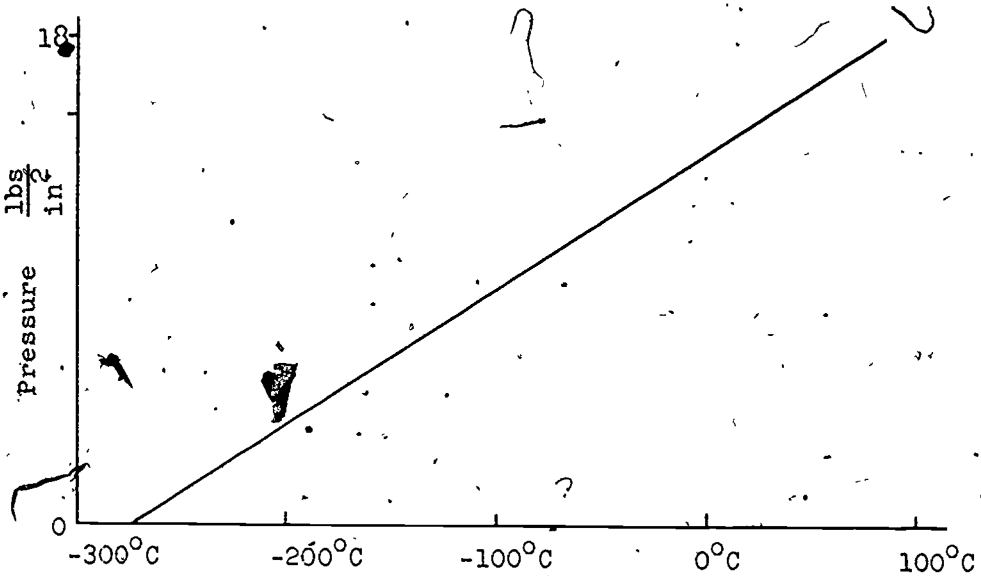


Figure 8

The "magic" temperature as determined with extreme care by research chemists in the past is very near -273°C. If you obtained anything between -260 and -285°C your work has been excellent! Surprisingly, some real gases almost behave in the manner suggested. It is almost, but not quite, legitimate to extend the temperature domain this far in these few cases. Most gases liquify first, and if not, other effects come into play which make low-temperature gases behave somewhat differently than high-temperature ones.

Exercise 3

1. The table below shows the speed of sound in air at various Fahrenheit temperatures. The absolute zero of temperature on the Fahrenheit scale is -460°F.

Temp. (°F)	-30	-20	0	20	50	80	110
Speed (ft/sec)	1030	1040	1060	1080	1110	1140	1170

(a) Draw a graph showing the relation between F and speed of sound (S). Make temperature values the domain and let the origin represent 1000 on the vertical axis.

(b) Write the equation for the relation between F and S.

2. The relation between Centigrade and Fahrenheit temperatures is expressed in the equation $C = \frac{5}{9}(F - 32)$.

Write the equation obtained by reversing the variables.

3. In an experiment on Gay-Lussac's Law, a student found that the pressure of the gas was 7.5 lb/sq in at 20°C and 9.5 lb/sq in at 100°C.
   (a) Graph the relation.
   (b) Write the equation representing the relation between the pressure (P) and the Centigrade temperature (t).
   (c) At what temperature would the pressure of the gas be 8.2 lb/sq in?
   (d) What would be the pressure of the gas at 50°C?

3.8 Graphical Translation of Coordinate Axes

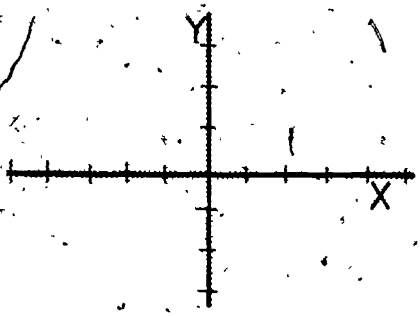
A non-vertical line drawn upon coordinate paper always represents some sort of linear function. The constants c, d and m in the point-slope representation locate the line. A second line will be described by different values of these constants. In other words, the fact that both lines really are just that -- lines -- is an important consideration. Do you suppose that the two lines really are the "same" somehow? Perhaps we have nothing more than two mathematical descriptions of a single line.

For many purposes it is very useful to think of all lines that can be drawn as different positions of a single line. It is only the mathematical description of the line that differs. One point of view would be to think of the line as having moved from one position to another with respect to the coordinate axes. On the other hand, we may also think of the coordinate axes as having shifted with respect to the line. Either viewpoint is as good as the other, but in the discussion that follows we will always consider motion of the coordinate axes with respect to the graph of the function.

We have, then, a method for changing the description of a line simply by moving the coordinate axes with respect to that line.

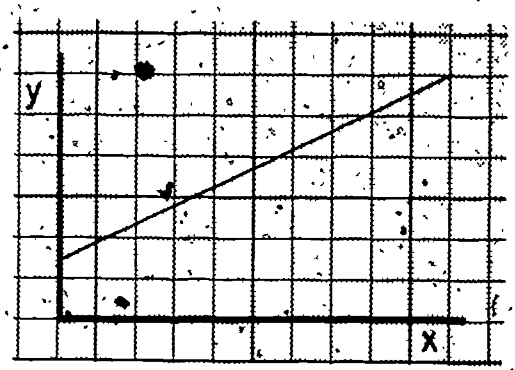
An $8\frac{1}{2} \times 11$ - inch sheet of frosted acetate provides an excellent surface upon which a set of movable coordinate axes may be drawn. These "moving" axes must carry the same scale as those on your coordinate paper. When this plastic sheet is placed upon a regular sheet of coordinate paper that carries a graph of some sort, the graph beneath is easily seen through the frosted acetate. In this way the graph can be readily related to the "new" coordinate axes carried by the overlying plastic sheet. These new axes may be positioned in any manner whatsoever.

The sheet of frosted acetate, the coordinate paper and graph, and the stack formed by the two are illustrated in Figure 9(a), (b) and (c).



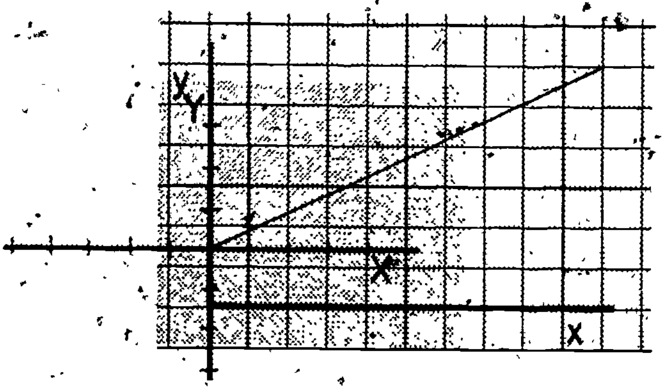
Acetate sheet with axes.

(a)



Coordinate paper and graph.

(b)



Graph viewed in relation to new axes X and Y.

(c)

Figure 9

Figure 9(c) shows the coordinate axes X and Y displaced upward with respect to the origin of the graph beneath. The frosted side of the acetate is up, for upon this surface pencil lines can be drawn which are easily erased.

It must be realized that if we are to allow any kind of motion of the coordinate axes X and Y whatsoever, this motion might be rather complicated. Matters can be simplified, however, by recognizing that any complex motion may be broken into two parts. One of these parts is simple straight-line motion, and the second is rotation. In other words, the axes X and Y may be displaced along a line, rotated without straight-line motion, or a combination of these two types of motion may be used. Only straight-line, or linear, motion of these axes will be considered. This motion will keep horizontal lines horizontal and vertical lines vertical.

There is one other important point to be made. Any motion of translation only can be considered as made up of two translations, one in the horizontal direction and one in the vertical.

Suppose we start with the X and Y axes on the plastic overlay coincident with the x and y axes on the sheet underneath. The use of the capital letters X and Y on the overlay will help us to remember that these represent the axes that are moved, or "translated". When these axes are translated, the entire plastic sheet moves horizontally and vertically and is not rotated. The X axis must always remain parallel to the original x axis, and the Y axis must always remain parallel to the original y axis. It is a simple matter to guarantee that no rotation has been involved in the motion of the axes by maintaining the X and Y axes at all times parallel to the ruled lines of the coordinate paper underneath.

Figure 9(c) suggests one of the many ways in which the coordinate axes may be shifted. The axes have been moved upward until the new origin is coincident with the original y-intercept. Using this new position of the axes, the equation of the line would now be of the form $Y = mX$, where before it was of the form $y - d = m(x - c)$. Notice that the slope of a line never changes as the axes are translated. This is an extremely important feature of a linear translation -- a feature that is not to be found when coordinate axes are rotated.

The central idea underlying the concept of the translation of axes is simply this: It makes no real difference where the coordinate axes are placed on a sheet of coordinate paper, for they may always be moved to a new position through horizontal motion, vertical motion, or both. The placement of axes is a purely arbitrary matter, and in practice, they are placed in one position or another strictly as a matter of convenience. A given physical situation usually suggests a "natural" location for these axes.

We now wish to translate the axes for a particular physical situation, namely, that of the Gay-Lussac's Law experiment. Refer to your own graph of the temperature - pressure relation similar to that shown in Figure 8. It is important to realize that Figure 8 shows the horizontal axis, but not the vertical axis. The vertical line at the left side is merely a pressure scale. and not an axis.

If we now wish to translate these coordinate axes, one could well ask: Where do we move them and why? There seem to be two logical possibilities. We might move the origin either to the point where the graph intersects the (old) y-axis, or to the point where it intersects the (old) x-axis. The first intercept corresponds to the 0^oC point, while the second corresponds to the zero pressure point. The temperature can fall below 0^oC but the pressure cannot fall below zero. This fact seems to make the horizontal axis intercept (zero pressure) the more interesting of the two.

The desired translation of the axes, then, is indicated in Figure 10.

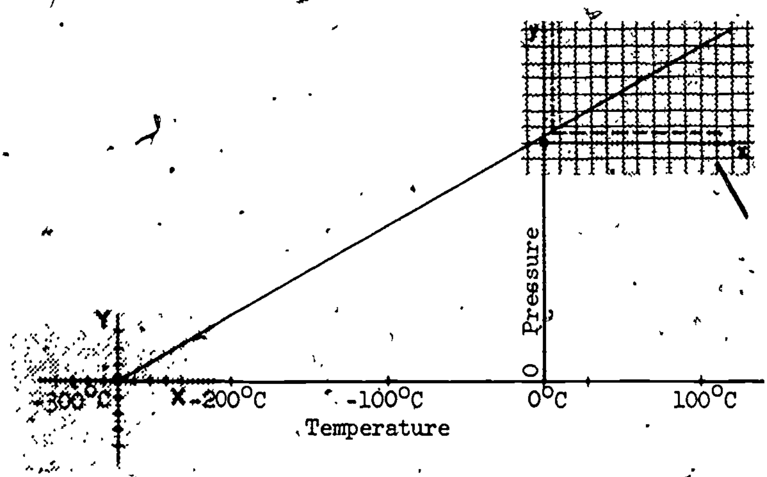


Figure 10

In the figure, the two heavy dots represent the initial and final position of the axes. x and y label the initial positions of the axes, and X and Y label the translated axes. The dashed lines indicate the extent of your original graph.

It is important to realize that both coordinate axes have been shifted here. One might be tempted to say that only the vertical axis has moved since the horizontal axis remains the same. This is not true because the movement of the horizontal axis carries with it the zero point which must always lie at the origin. Scales are not shown along the new axes in Figure 10, but (0,0) must lie at the intersection of the translated axes X and Y. In short, there is no such thing as translating one axis without translating the other.

For the translated axes in this example, we have established a new zero for a temperature scale located at the new origin. The zero of pressure has remained unchanged. The new zero of temperature is the temperature at which the pressure in a gas would also be zero. This new temperature scale is so important in both chemistry and physics that it is given a new name, the Absolute temperature scale. Temperatures in this scale are indicated by writing $^{\circ}$K. As indicated previously, this point on the Centigrade scale falls approximately at -273°C. Since the size of an Absolute degree is the same as the size of a Centigrade degree, we find a simple relation between $^{\circ}$C and $^{\circ}$K, namely $^{\circ}C + 273 = {}^{\circ}K$.

To summarize, we have found that the translation of coordinate axes may be accomplished easily using a transparent overlay of frosted acetate. The new axes may be placed anywhere so long as the translated axes remain parallel to their original position at all times. With the axes in any new position, the graph may easily be interpreted with respect to the shifted axes to arrive at a new description of the graph. This is done visually without fuss or bother. We see that in this way of doing things, it is only the mathematical description of the graph that changes as the axes are translated, not the graph itself.

Exercise 4

1. Find the load-position graph that you drew in the Loaded Beam experiment. Using a sheet of frosted acetate that carries coordinate axes X and Y, translate the origin on the overlay to the y-intercept on your graph. What is the equation of your "best" straight line with respect to the shifted axes?

2. How could you perform the Loaded Beam experiment to obtain the equation found in Exercise 1 directly?

3. Draw the line in the first quadrant that contains the point (2,3) and whose slope is $\frac{1}{2}$. Use your plastic overlay to obtain the new equations of this line when the origin is shifted

(a) to the y-intercept;

(b) to the left 3 units;

(c) to the right 4 units and up 3 units.

4. Draw the line in the first quadrant which contains the points (1,7) and (7,5). Use your plastic overlay to obtain the new equations of this line when the origin is shifted

(a) to the x-intercept;

(b) to the y-intercept;

(c) to the point (4,6).

3.9 Algebraic Translation of Coordinate Axes

Although the mathematical description of a graph may be obtained easily by the graphical procedure described in the preceding section, it is also desirable to be able to describe a graph after the axes have been translated without resorting to the analysis of the graph itself.

We will discuss linear functions only, and for this purpose we will use the so-called point-slope representation of a line.

First it will be shown that the point-slope representation of a line can be considered as one in which the coordinate axes have already been translated in both the horizontal and vertical directions.

Suppose we start with a line that runs through the origin as in Figure 11.

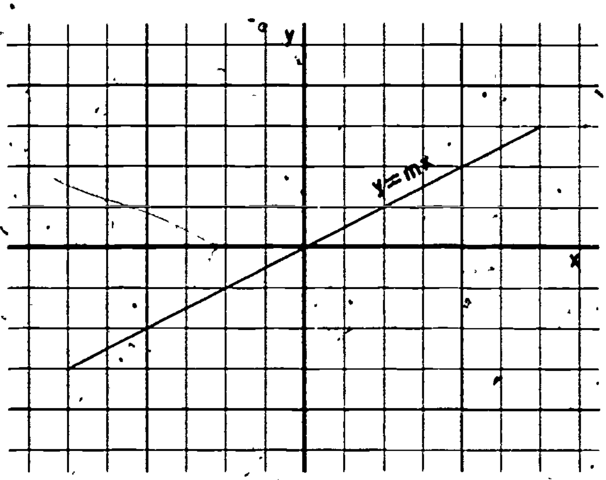


Figure 11

Let us now translate the coordinate axes x and y both to the left and downward. These shifted axes are denoted, as before, X and Y. This translation is shown in Figure 12.

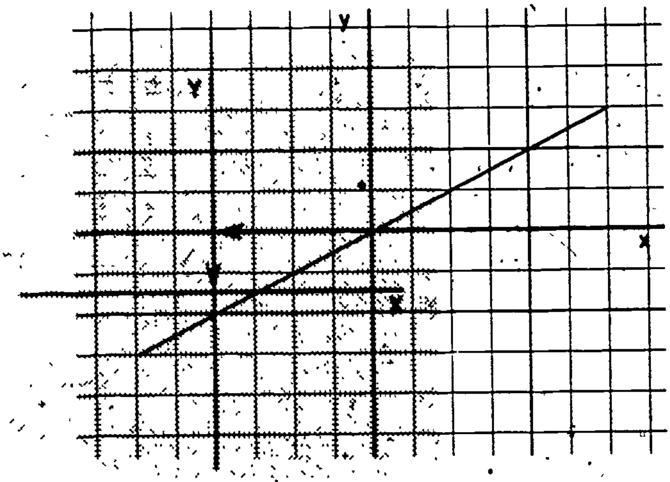


Figure 12

Since the point (c,d) is a particular point on the line, we can now describe the line in the familiar point-slope form as

$$Y - d = m(X - c).$$

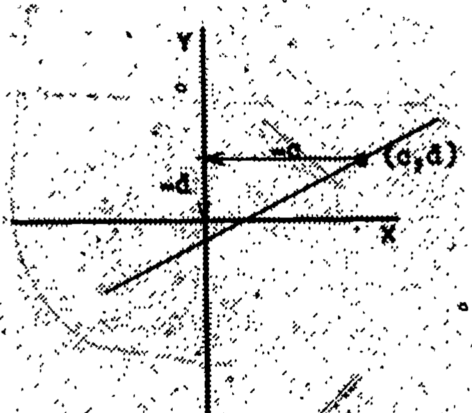


Figure 13

But if we now write this same expression in slightly different form

$$Y + (-d) = m[X + (-c)] \tag{1}$$

we may draw a remarkable conclusion. Since the quantities in parentheses are the horizontal and vertical translation distances, this equation tells us what the point-slope representation of a line is by setting the Y-coordinate plus the vertical translation equal to the slope of the line times the quantity, X-coordinate plus the horizontal translation. The translations involved are those that carry the origin from a point on the line to a point off the line.

Equation (1) is perfectly general, for we could just as well have moved the origin from any point to any other point. Consider a second translation of the axes as indicated in Figure 14.

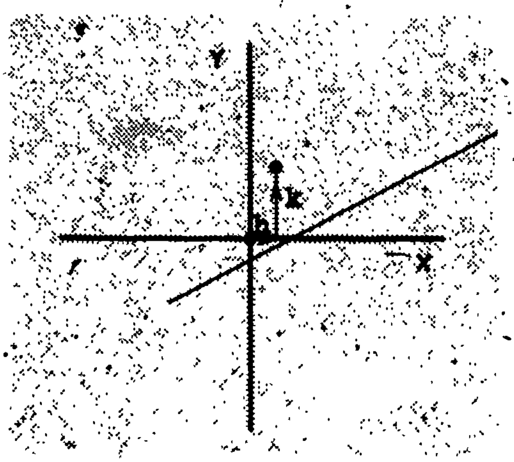


Figure 14

This time let the horizontal translation be designated h and the vertical translation be the symbol k. Both of these quantities as shown in Figure 14 are positive. Since this is a second translation, all we have to do is add this second horizontal translation to the first one, and add this second vertical translation to the first vertical translation. Doing this gives us

$$Y + (-d) + k = m[X + (-c) + h]. \tag{2}$$

Notice that Equation (2) is of exactly the same form as Equation (1). In both cases we have

$$Y + (\text{vertical translation}) = m[X + (\text{horizontal translation})].$$

In each case the translations are the total vertical and horizontal translations starting with the origin of the coordinate axes on the line.

Equation (2) can now be used to represent the new mathematical description of a line that results from a translation of axes from any previous point whatsoever. If the original description of the line was $Y - d = m(X - c)$, Equation (2) gives the new expression using the two old constants c and d and, in addition, inserting two new ones h and k to represent the horizontal and vertical translation distances respectively.

This final Equation (2) represents analytically the same new description of a line that was previously obtained using graphical analysis with the frosted acetate sheet.

Exercise 5

1. When we extended the temperature domain for the Gay-Lussac's Law experiment, we found that the graph intercepted the temperature axis near the (-273,0) point. Algebraically translate the origin of your graph to this intercept. Write the new equation of the line. What are the new units of temperature, pressure, and the slope of the line?

2. Draw the line in the first quadrant that contains the point (2,3) and whose slope is $\frac{1}{2}$. Write the equation of this line in point-slope form. Obtain the equation of this line algebraically when the origin has been translated

   (a) to the y-intercept;

   (b) to the left 3 units;

   (c) to the right 4 units and up 3 units.

   Compare your results to those obtained graphically in Exercise 3 of the previous section.

3. Draw the line in the first quadrant which contains the points (1,7) and (7,5). Write the equation of this line in point-slope form. Obtain the equation of this line algebraically when the origin has been translated

   (a) to the x-intercept;

   (b) to the y-intercept;

   (c) to the point (4,6).

   Compare your results to those obtained graphically in Exercise 4 of the previous section.

3.10 Summary

In this chapter we considered the problem of determining the behavior of an object bouncing on a trampoline. An experimental trampoline was set up, consisting of a marble bouncing on a balloon stretched over a pie plate. The heights of the successive bounces were a function of the bounce number and, thus, were an example of a function defined only on the integers.

The data which resulted from this experiment did not exhibit a linear relationship, but we found that the graph of the height of a bounce plotted against the height of the previous bounce was linear.

Next, we considered the experimental relation between the temperature of a gas and its pressure. This data also gave a linear relation. We considered the possibility of extrapolating this linear relation to temperatures below those actually used and found that there was a limit below which we could not go without reaching negative pressures. The resulting linear relation could be simplified by changing the position of the coordinate axes.

The chapter closed with a discussion of the geometric and algebraic properties of translations.

Appendix A

GRAPHING EXPERIMENTAL DATA

A.1 The Location of Points in a Plane

There are a great many instances where a certain place can be located by the use of pairs of numbers. Road maps frequently are divided into small blocks by a series of horizontal and vertical lines (Figure 1). The vertical lines are then designated by the numbers and the horizontal lines by letters. A motorist can then locate a certain city by referring to a table which will tell him that the city is (approximately) at the intersection of "line D" and "line 5". Theatre tickets have the row number and seat number printed on them so a person can find his seat.

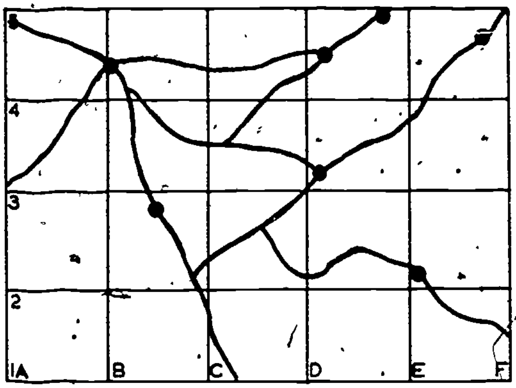


Figure 1

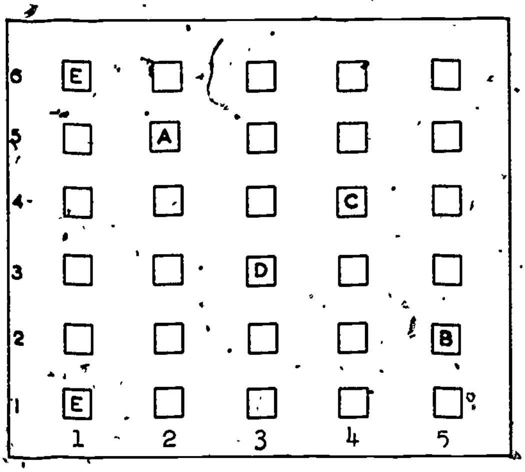


Figure 2

A convenient way of referring to a particular seat in a classroom where the chairs are located in rows is based on assigning numbers to each chair. A familiar pattern for seating in a classroom involves five rows of six chairs each (Figure 2). We can refer to a particular seat by naming the row and then the chair number in the row. In our diagram we show five rows and seats one through six in each row. Seat A is "row 2, seat 5" while seat B is "row 5, seat 2". If we agree to refer to a particular seat by first naming the row and then the chair number in the row, seat A can be indicated as seat (2,5) while seat B would be seat (5,2).

Each of these pairs of numbers is called an <u>ordered pair</u>. Two numbers are needed to locate the particular seat, and the order of reporting the two numbers is extremely important. What ordered pairs would you associate with points C, D, E and F in Figure 2?

In newspapers and magazines, as well as in textbooks, a graph of the type shown in Figure 3 is often used to present data.

In collecting the data for this graph you would have to use pairs of numbers, one number for the row, and one for the number of protractors needed. If we agree to state these two numbers in a certain order, such as (row number, number of protractors) we can represent our data as a set of ordered pairs; {(1,4), (2,3), (3,6), (4,2), (5,5)}.

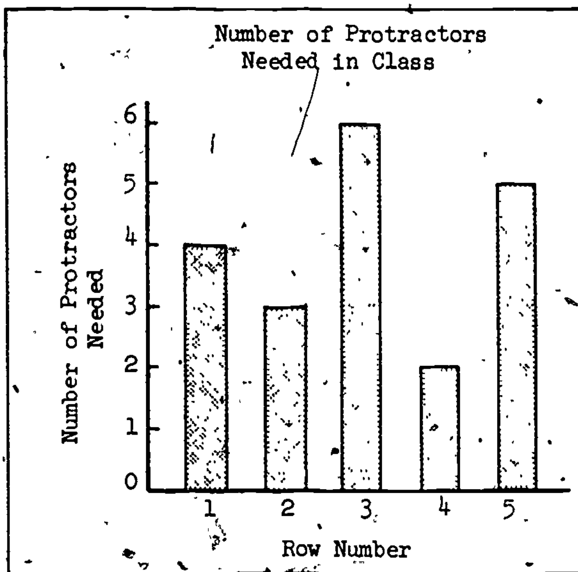


Figure 3

This notation using braces ({}) is formal mathematics notation for a set of ordered pairs. You probably would not record the information in this manner, however. Most likely you would simply make a table like the one shown below.

Row Numbers	1	2	3	4	5
Number of Protractors	4	3	6	2	5

In either case you have collected data and recorded it as ordered pairs.

A bar graph, such as that shown in Figure 3 is very useful for presenting numerical information in a clear and compact way, but the "bars" in this graph are not really necessary. We could just as well use dots on a sheet of graph paper to represent our ordered pairs. (Figure 4.)

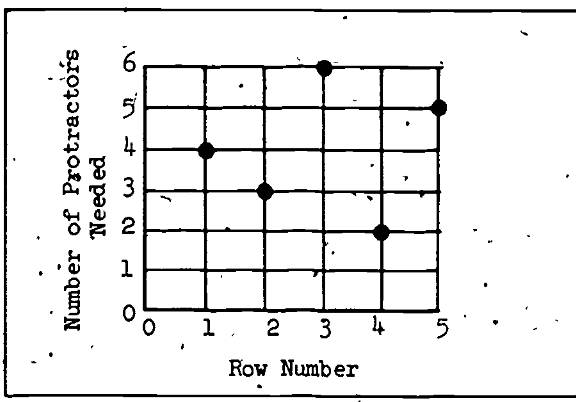


Figure 4

Scientists often make a graph of the observations obtained through experimentation. In the first chapter you were asked to complete the following table (Exercise 1) for a seesaw experiment. All of the data is given in this table.

Mass	12	2	8	24	16	6	3
Distance	4	24	6	2	3	8	16

Another way of listing this data is in set notation: {(12,4), (2,24), (8,6), (24,2), (16,3), (6,8), (3,16)}. This set of ordered pairs can be graphed in the same way as the previous example. We again use a piece of graph paper and begin with two perpendicular lines called axes. We can label the horizontal axis "mass in grams" and the vertical axis "distance in cm".

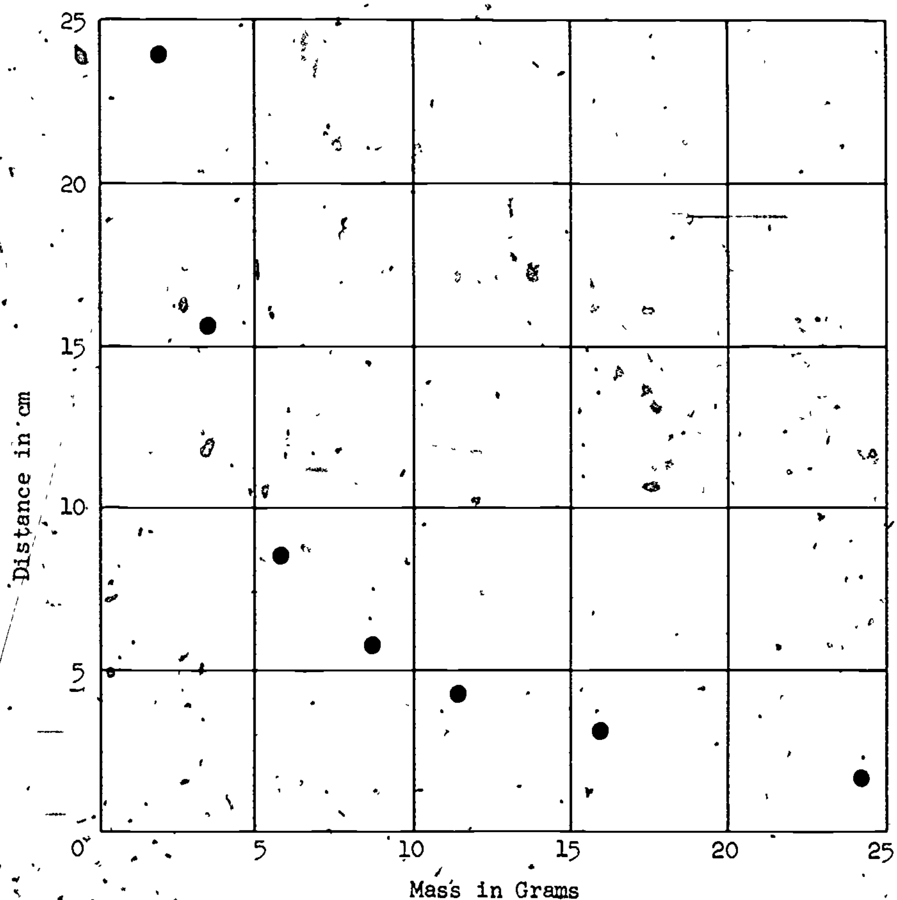


Figure 5

A.2 Coordinates

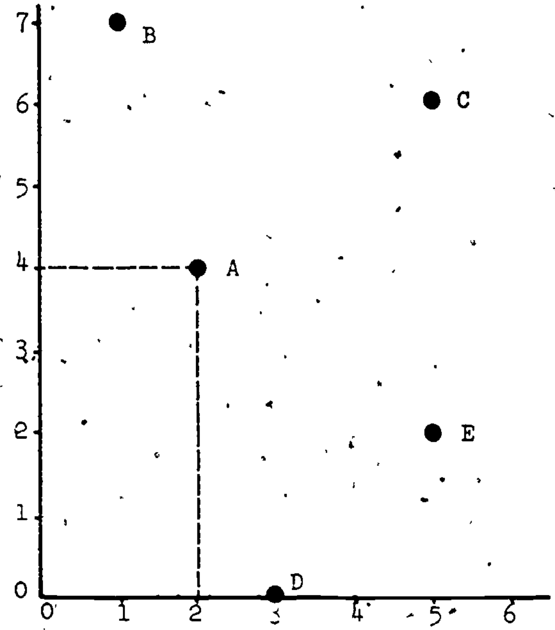


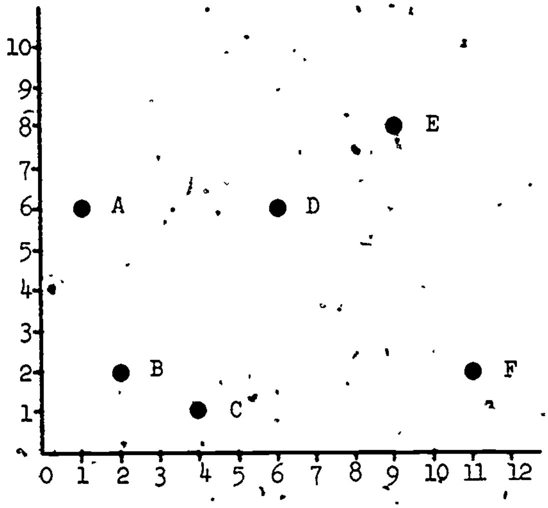
Figure 6

As we have seen, a set of ordered pairs of numbers can be represented by a graph. Each point on the graph represents one member of the set of ordered pairs. For example, point A in Figure 5 could be represented by either the pair (2,4) or the pair (4,2). As you have probably noted, we needed to make some decision as to the meaning of each member of the pair. The individual numbers of the ordered pair are called the coordinates of the point. The member of the ordered pair which indicates how far to the right of the zero point the point is located is called the horizontal coordinate of the point. The member of the ordered pair which indicates how far above the horizontal axis the point is located is called the vertical coordinate of the point.

It is common practice in preparing graphs, to arrange ordered pairs so that the first member of the pair represents the horizontal coordinate and the second member represents the vertical coordinate. Using this convention, point A in Figure 6 will have as its coordinates the ordered pair (2,4) rather than (4,2). Point B is described by the ordered pair (1,7). Can you write the coordinates of points C, D and E?

Exercise 1

1. Write the ordered pairs of numbers which are associated with the points A through F in the figure below.



2. Graph the following sets of ordered pairs on the same sheet of graph paper.

   (a) $\{(0,0), (1,1), (2,2), (3,3), (4,4), (5,5)\}$
   (b) $\{(0,0), (1,2), (2,4), (3,6), (4,8), (5,10)\}$
   (c) $\{(0,0), (1,3), (2,6), (3,9), (4,12), (5,15)\}$
   (d) $\{(0,0), (1,\frac{1}{2}), (2,1), (3,1\frac{1}{2}), (4,2), (5,2\frac{1}{2})\}$
   (e) $\{(0,0), (1,1), (2,4), (3,9), (4,16), (5,25)\}$

3. Make a set of at least five ordered pairs to satisfy the following conditions.

   (a) The ordered pairs for which the vertical coordinate is 6 times the horizontal coordinate.
   (b) The ordered pairs for which the vertical coordinate is 3 times the horizontal coordinate.
   (c) The ordered pairs for which the vertical coordinate is 2 more than twice the horizontal coordinate.
   (d) The ordered pairs for which the vertical coordinate is the square root of the horizontal coordinate.
   (e) The ordered pairs for which the vertical coordinate is the cube of the horizontal coordinate.

4. Make a graph of the data recorded in each of the tables below. In each case, the top row indicates the horizontal coordinates, and the bottom row the vertical coordinates. Be sure to label the axes correctly (refer to Figure 5 in text).

(a)

Time (sec)	0	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	3
Speed (meters/sec)	0	$\frac{1}{2}$	2	$4\frac{1}{2}$	8	18

(b)

Distance of object (cm)	30	15	10	6	5	3	2	1
Distance of image (cm)	1	2	3	5	6	10	15	30

(c)

Weight in oz	$\frac{1}{2}$	1	$1\frac{1}{2}$	2	$2\frac{1}{2}$	3	$3\frac{1}{2}$	4
Cost in cents	5	5	5	10	10	15	15	20

Appendix B

SCIENTIFIC NOTATION

B.1 Bases and Exponents

Many of the measurements in the physical sciences yield numbers which are either extremely large or extremely small. For example, the speed of light is approximately 300,000,000 meters per second, and the radius of the helium atom nucleus is approximately 0.0000000000024 cm. Some method of writing such numbers is needed which will make it relatively easy to compare and work with these numbers. To introduce such a system it is important that we first develop the necessary concepts and symbols.

The number 625 can be represented as the product of four fives, that is

$$625 = 5 \times 5 \times 5 \times 5 .$$

It is often convenient to think of 625 as "four fives multiplied together", but this type of notation is somewhat inconvenient because it is so lengthy. You probably already know that 3×3 can be written as 3^2 (three squared). The "3" indicates that we are going to multiply 3's together, and the "2" that we are going to multiply <u>two</u> of them. If we extend this notation to a product such as $5 \times 5 \times 5 \times 5$ we can write it as 5^4. The "5" means that we are going to multiply 5's together, and the "4" means that we are going to multiply <u>four</u> of them. Numbers written in this manner are called <u>powers</u>. For example, 5^4 is the fourth power of five. In a similar manner 9^3, the cube of 9, means $9 \times 9 \times 9$, and 10^5, the fifth power of 10, means $10 \times 10 \times 10 \times 10 \times 10$.

In an expression like 5^4, the number which is to be multiplied (in this case, "5") is called the <u>base</u>; the "4" which indicates how many 5's we are going to multiply is called an <u>exponent</u>. In 9^3 the base is "9" and the exponent is "3". How can you write the expression $2 \times 2 \times 2 \times 2 \times 2$ using exponents? What is the base? What is the exponent? How would you read such a number?

The number 288 can be written as $2 \times 2 \times 2 \times 2 \times 2 \times 3 \times 3$. Using the associative property of multiplication,

$$288 = (2 \times 2 \times 2 \times 2 \times 2) \times (3 \times 3)$$

or in exponent form

$$288 = 2^5 \times 3^2 .$$

This expression would be read as "two to the fifth power times three squared".

Exercise 1

1. For each of the following, indicate the base and the exponent.

   (a) 6^3 (d) 9^2

   (b) 10^5 (e) x^2

   (c) 5^8 (f) x^5

2. Using exponents, write each of the following in briefer form.

   (a) $3 \times 3 \times 3 \times 3$

   (b) $10 \times 10 \times 10$

   (c) $3 \times 3 \times 3 \times 5 \times 5$

   (d) $5 \times 3 \times 2 \times 2 \times 3 \times 5$

   (e) $1.25 \times 10 \times 10 \times 10$

3. What is the value of each of the following?

   (a) 3^4 (e) 10^6

   (b) 2^3 (f) $3^2 + 2^3$

   (c) 9^2 (g) $3^3 + 2^2$

   (d) 5^4

In problems 4 - 7, tell which statements are true and which are false.

Example:

$$2^3 + 3^3 = 5^3 \text{ ?}$$

$$\begin{aligned} 2^3 + 3^3 &= (2 \times 2 \times 2) + (3 \times 3 \times 3) \\ &= 8 + 27 \\ &= 35 , \end{aligned}$$

$$\begin{aligned} 5^3 &= 5 \times 5 \times 5 \\ &= 125 . \end{aligned}$$

35 is not equal to 125, hence the equation is <u>false</u>.

4. $2^3 \times 3^3 = 6^3$

5. $2^3 \times 2^3 = 2^6$

6. $3 \times 3^3 = 9^3$

7. $2^5 - 2^3 = 2^2$

B.2 Powers of Ten

Our decimal system of writing numerals is based on the number ten. Starting at the units place, each place to the left has a value ten times as large as the place to the right.

Hundred thousands	Ten thousands	Thousands	Hundreds	Tens	Units
10×10×10×10×10	10×10×10×10	10×10×10	10×10	10	1

These numbers can be written using exponents:

$$100{,}000 = 10 \times 10 \times 10 \times 10 \times 10 = 10^5$$
$$10{,}000 = 10 \times 10 \times 10 \times 10 = 10^4$$
$$1{,}000 = 10 \times 10 \times 10 = 10^3$$
$$100 = 10 \times 10 = 10^2$$
$$10 = 10 = 10^1$$

In the above table each succeeding number is $\frac{1}{10}$ of the previous number, and thus each exponent is one less than the previous one. In order to complete the table the next number should be 1 and the next exponential form should be 10^0.

This pattern repeats itself for powers other than powers of ten.

Powers of two	Powers of three	Powers of four
$16 = 2 \times 2 \times 2 \times 2 = 2^4$	$81 = 3 \times 3 \times 3 \times 3 = 3^4$	$256 = 4 \times 4 \times 4 \times 4 = 4^4$
$8 = 2 \times 2 \times 2 = 2^3$	$27 = 3 \times 3 \times 3 = 3^3$	$64 = 4 \times 4 \times 4 = 4^3$
$4 = 2 \times 2 = 2^2$	$9 = 3 \times 3 = 3^2$	$16 = 4 \times 4 = 4^2$
$2 = 2 = 2^1$	$3 = 3 = 3^1$	$4 = 4 = 4^1$
Each number is $\frac{1}{2}$ of the previous number.	Each number is $\frac{1}{3}$ of the previous number.	Each number is $\frac{1}{4}$ of the previous number.

In each case the next number will be 1, and the next exponential form will be the common base with a zero exponent.

$2^0 = 1$ $\qquad 3^0 = 1$ $\qquad 4^0 = 1$

From this we can make the following definition:

For every number B not equal to zero

$$B^0 = 1 .$$

The expression 0^0 is called an indeterminant form and has no meaning.

B.3 Negative Exponents

We can also use exponents to express numbers which are less than one. In particular, we can express decimal fractions in terms of powers of ten.

$$.1 = \frac{1}{10} = \frac{1}{10^1}$$

$$.01 = \frac{1}{100} = \frac{1}{10^2}$$

$$.001 = \frac{1}{1000} = \frac{1}{10^3}$$

etc.

In order to simplify these expressions we are forced to make another definition. In mathematics we usually write a fraction which has some power of a number in the denominator in terms of a "negative" exponent.

$$\boxed{\frac{1}{B^n} = B^{-n} \qquad (\text{B not equal to zero})}$$

In this way $\frac{1}{10^3}$ becomes 10^{-3}; $\frac{1}{10^4}$ is 10^{-4}, etc. Zero with a negative exponent implies division by zero which is not defined.

In a later course, when you study the various operations which can be performed with exponents, you will learn how these definitions have come about.

B.4 Scientific Notation

Scientists have used the type of notation introduced in the previous sections to develop a method of writing extremely large or expremely small numbers. This method is called <u>scientific notation</u>, and allows us to express numbers as the product of a number between one and ten and some power of ten. If you think back to the relation between multiplying or dividing by ten and use your knowledge of the decimal system, you will be able to see how the product of some integral power of ten with a number between one and ten can be used to represent any number.

$$1.23 \times 10 = 12.3 = 1.23 \times 10^1$$

$$1.23 \times 10 \times 10 = 123 = 1.23 \times 10^2$$

$$1.23 \times 10 \times 10 \times 10 = 1230 = 1.23 \times 10^3$$

Now try the following problems:

$$1.23 \times \frac{1}{10} = .123 = 1.23 \times 10^{-1}$$

$$1.23 \times \frac{1}{100} = .0123 = 1.23 \times 10^{-2}$$

$$1.23 \times \frac{1}{1000} = .00123 = 1.23 \times 10^{-3}$$

These examples and problems lead us to the following general rule.

> Let a number be given in decimal form, and suppose we wish to multiply this number by some power of 10. To do this we merely need to move the decimal point the same number of places as the exponent of 10, to the right for positive exponents, and to the left for negative exponents.

We can use what we have just learned to simplify measurements such as those mentioned in Section B.1 . For example, the speed of light is approximately 300,000,000 meters per second. This can be written as

$$3 \times 100,000,000 \text{ meters per second}$$

or, using exponents,

$$3 \times 10^{8} \text{ meters per second} .$$

The radius of the nucleus of the helium atom is approximately

$$0.00000000000024 \text{ cm}$$

which can be written as

$$2.4 \times .0000000000001 \text{ cm}$$

which equals

$$2.4 \times 10^{-13} \text{cm} .$$

This type of operation is called expressing measurements in scientific notation. Here are some examples. Notice that in each case the measurements are expressed as some number between one and ten multiplied by some power of ten. Notice also that when you write a number in scientific notation, the exponent of ten will be positive if the number is larger than ten, or negative if the number is smaller than one. The size of the exponent is the number of places the decimal point must be moved to bring it directly after the first nonzero digit.

$$2540 \text{ mm} = 2.540 \times 1000 \text{ mm} = 2.540 \times 10^{3} \text{ mm}$$

$$93,000,000 \text{ miles} = 9.3 \times 10,000,000 \text{ miles} = 9.3 \times 10^{7} \text{ miles}$$

$$0.0683 \text{ meters} = 6.83 \times 0.01 \text{ meters} = 6.83 \times 10^{-2} \text{ meters}$$

$$0.00008215 \text{ inches} = 8.215 \times 0.00001 \text{ inches} = 8.215 \times 10^{-5} \text{ inches}$$

Example 1: Write 978.23 in scientific notation.

To change the number 978.23 to 9.7823 (a number between one and ten) the decimal place must be moved two places to the left. To restore 9.7823 to the original form would require moving the decimal point two places to the right. We have found that this can be done by multiplying by 10^2. Therefore we can make the statement:

$$978.23 = 9.7823 \times 10^2.$$

Example 2: Write 0.0034 in scientific notation.

To change 0.0034 to 3.4 (a number between one and ten) the decimal point must be moved three places to the right. To change 3.4 to its original form would require moving the decimal point three places to the left. We have found that that can be done by multiplying by 10^{-3}. Therefore

$$0.0034 = 3.4 \times 10^{-3}$$

Exercise 2

1. Perform the indicated multiplications mentally and write your answers.

   Example: $26.3 \times 10^{-2} = .263$

   (a) $259.4 \times 10^{-4} =$

   (b) $3.258 \times 10^{2} =$

   (c) $.023 \times 10^{3} =$

   (d) $35.68 \times 10^{-1} =$

   (e) $358.2 \times 10^{-3} =$

   ~~(f) $151 \times 10^{-4} =$~~

   (g) $.0031 \times 10^{5} =$

   (h) $29.35 \times 10^{-2} =$

   (i) $3.05 \times 10^{-6} =$

   (j) $3.05 \times 10^{6} =$

2. Express these measurements in scientific notation.

   (a) There are more than 4,500,000 red corpuscles per cubic mm of blood.

   (b) and (c) If a given sample of material contains 2,000,000 atoms of U_{238} in 1964, this same sample will contain 250,000 atoms of U_{238} in the year 13,500,001,964. (Write the numbers of atoms in scientific notation.)

   (d) The normal concentration of glucose in the human cell is .0007 .

   (e) The distance to the sun is 150,000,000 km .

Appendix C

METRIC SYSTEM

C.1 Metric Prefixes

The definition of measurement states that it is a process in which the object or event to be measured is compared to the standard unit for the object or event. The process of measurement of physical quantities begins with the establishment of three primary standards, one for length, one for mass, and one for time. Two different observers will obtain the same result only if they have agreed to use the standards. Since length, mass, and time cannot be defined, the measurement process for such quantities must be established by agreement. Certain fundamental units have been established by custom, by national legislation, and by international agreement.

The most widely used system throughout the world is the metric system. Except for the English-speaking countries, this is the system which is in general use in all major countries. It is also the system used for scientific work in all countries.

The metric system consists of a set of basic units originally established by the French Academy of Science after the French Revolution. This system is a decimal system and certain prefixes are used with the basic units to give new units. The prefixes are based on powers of ten as shown in the table below.

Prefix		Symbol		Value				
micro	=	(μ)	=	one millionth	=	0.000001	=	10^{-6}
milli	=	(m)	=	one thousandth	=	0.001	=	10^{-3}
centi	=	(c)	=	one hundredth	=	0.01	=	10^{-2}
deci	=	(d)	=	one tenth	=	0.1	=	10^{-1}
BASIC UNIT	=		=	one	=	1	=	10^{0}
deka	=	(dk)	=	ten	=	10	=	10^{1}
hecto	=	(h)	=	one hundred	=	100	=	10^{2}
kilo	=	(k)	=	one thousand	=	1000	=	10^{3}
mega	=	(M)	=	one million	=	1,000,000	=	10^{6}

The basic units used with these prefixes are

meter - length
gram - mass
second - time

Some examples of the various combinations are shown below.

1 kilometer (km) equals 100 meters.

1 milligram (mg) equals $\frac{1}{1000}$ gram.

1 centimeter (cm) equals $\frac{1}{100}$ meter.

1 hectogram (hg) equals 100 grams.

Not all of the possible combinations are actually used. A scientist would have little use for a unit such as a hectosecond. The table below lists some of the combinations which are generally used.

Length	Mass	Time
micron	microgram	microsecond
millimeter	milligram	millisecond
centimeter		
meter	gram	second
kilometer	kilogram	

Notice the first entry in this table. Instead of micrometer, we write micron. The special name is used because this unit is in very commun use in certain fields and a shorter term is valuable. The word micrometer is also used for another purpose. (Do you know what? If not, look it up.) The following examples will introduce you to the process of conversion.

These prefixes are also used with other units, such as kilovolts and microfarads in electricity. Have you ever heard of a megaton? In the English system of units we have various units of volume, cubic feet, quarts (both dry and liquid), etc. In the metric system the unit of volume is the <u>liter</u>. The liter is approximately equal to 1000 cubic centimeters (actually 1 liter equals 1000.028 cubic centimeters). The prefixes are used with this unit to form new units such as <u>milliliter</u>, <u>deciliter</u>, etc.

C.2 Conversion of Units

There are many times when it is necessary or convenient to change from one basic unit of measure to another of the same nature, such as from centimeters to meters. The process of changing from one unit to another without actually going through the process of measuring with the new unit is called "conversion of units". Since, by definition

$$1 \text{ mm} = \frac{1}{1000} \text{ m},$$

we can also say that

$$1000 \text{ mm} = 1 \text{ m}.$$

In a similar way

$$1 \text{ meter} = 10 \text{ decimeters} = 100 \text{ centimeters} = 1000 \text{ millimeters}.$$

With a little manipulation with numbers we can arrive at the following:

Length	Mass	Volume
10 mm = 1 cm	10 mg = 1 cg	10 ml = 1 cl
10 cm = 1 dm	10 cg = 1 dg	10 cl = 1 dl
10 dm = 1 m	10 dg = 1 g	10 dl = 1 liter
⌠	⌠	etc.
1000 m = 1 km	1000 g = 1 kg	
etc.	etc.	

Example 1:

Suppose, for example, we have a measurement of 1253 mm. To express this measurement in centimeters, we note that

$$10 \text{ mm} = 1 \text{ cm} \tag{1}$$

or,

$$1 \text{ mm} = \frac{1}{10} \text{ cm} \tag{2}$$

The expression 1253 mm can be thought of as 1253 millimeter units, or

$$1253 \ (1 \text{ mm}) . \tag{3}$$

The conversion to centimeters can be made by referring to (2) and replacing the (1 mm) with its equivalent, $(\frac{1}{10} \text{ cm})$.

Thus

$$1253 \text{ mm} = 1253(\frac{1}{10} \text{ cm}) = 125.3 \text{ cm} .$$

In a similar way, we could find the decimeter measure and the meter measure of this measurement. We begin with the fundamental relation between the units

$$1 \text{ m} = 10 \text{ dm} = 100 \text{ cm} = 1000 \text{ mm}$$

and rearrange it so that

$$1 \text{ mm} = \frac{1}{10} \text{ cm} = \frac{1}{100} \text{ dm} = \frac{1}{1000} \text{ m} .$$

Then the measurement becomes

$$1253 \text{ mm} = 125.3 \text{ cm} = 12.53 \text{ dm} = 1.253 \text{ m} .$$

Notice that the four measures are related to each other. One measure can be obtained from the other by multiplying or dividing by some multiple of ten. The measures differ only in the position of the decimal point.

Example 2:

Change 23.7 grams into decigrams, centigrams and milligrams.

23.7 gm = 23.7 (10 decigrams) = 237 decigrams
23.7 gm = 23.7 (100 centigrams) = 2370 centigrams
23.7 gm = 23.7 (1000 milligrams) = 23700 milligrams

Example 3:

Change 50 ml to liters.

$$50 \text{ ml} = 50 \left(\frac{1}{1000} \text{ liter}\right) = .050 \text{ liter}$$

Exercise 1

1. 100 m = ________ dm = ________ cm .
2. 37.2 mg = ________ g = ________ kg .
3. 281 liters = ________ ml .
4. 1285 cm = ________ mm = ________ m .
5. 0.155 gm = ________ mg = ________ dg .
6. Change 500 milowatts to watts.
7. Change 900 deciliters to liters.

8. The frequency of radio station WICH is 1340 kilocycles. Express the frequency in cycles.

9. Change 55 minutes to microseconds.

C.3 English System of Units

The fact that we use inches, pounds, quarts, etc., in our everyday measurements and the metric system in scientific work means that we will have to learn how to make conversions between these two systems.

Although it is not well known, the United States adopted (in 1893) the International Meter and International Kilogram as fundamental standards. Our customary units, the yard and the pound, are defined in terms of these standards.

The table below lists some of the commonly used equivalents between the two systems. These values have been rounded off to a convenient number of places.

1 meter is approximately the same as a length of 39.37 inches.
1 inch is exactly the same as a length of 2.54 centimeters.
1 pound is approximately the same as the weight of 454 grams.
1 liter is approximately the same as 1.06 liquid quarts.

Example 1:

Change 20 meters to feet.

$$20 \text{ meters} = 20\ (1 \text{ meter}) = 20\ (39.37 \text{ inches})$$
$$= 78.74 \text{ inches}$$

and since

$$12 \text{ inches} = 1 \text{ ft} \quad \text{or} \quad 1 \text{ inch} = \frac{1}{12} \text{ ft}$$
$$78.74 \text{ inches} = 78.74\ (\tfrac{1}{12} \text{ ft})$$
$$= 6.56 \text{ ft}.$$

Example 2:

What mass in kilograms would weigh 3 pounds?

$$3 \text{ pounds} = 3\ (454 \text{ grams})$$
$$= 1362 \text{ grams}$$

which, of course, is equal to

$$1.362 \text{ kg}.$$

Exercise 2

1. 3 ft = ________ cm .

2. 114 liters = ________ qts = ________ gal .

3. 27 meters = ________ yards = ________ cm .

4. 428 ml = ________ cubic centimeters.

5. 6.5 ft = ________ cm = ________ m .

6. What mass in grams would weigh 1.5 pounds?

7. What is the weight (in pounds) of a 7 kilogram mass?

8. Change 1 quart to liters.

9. Change 1 yard to meters.

10. What is the weight (in pounds) of a 1 kilogram mass?

GLOSSARY

Part II

ANALYTICALLY -- A result is obtained analytically when it is obtained by computation (as opposed to experimentation).

ASSOCIATIVE PROPERTY OF MULTIPLICATION -- When three numbers are to be multiplied in a stated order, the product is independent of the grouping. That is,

$$a \times (b \times c) = (a \times b) \times c$$

BASE -- When a numeral is given in exponential form, the number which is to be multiplied by itself is called the base. That is, 3^4 means $3 \times 3 \times 3 \times 3$ and 3 is the base.

CLOSED PHRASE -- A closed phrase is a phrase which represents a specific number.

COINCIDENT -- Identical; having all points in common.

COORDINATE AXES -- Intersecting lines used to locate points in the plane by means of coordinates measured along the lines.

COORDINATES ON A PLANE -- The numbers associated, as an ordered pair, with a point of the plane are called the coordinates of the point.

DEFLECTION -- The amount of bend (as indicated by a pointer relative to a fixed scale).

DISPLACE -- When a directed movement of a coordinate axis is made, we say that the axis is displaced.

DISTRIBUTIVE PROPERTY -- If, in a given mathematical system, it is always true that $a \times (b + c) = (a \times b) + (a \times c)$, where a, b and c are any elements of the system, then we say that the given system has the distributive property. This is the distributive property of multiplication over addition.

DOMAIN -- The domain is the set of first elements of the ordered pairs in a relation or function.

ELEMENT -- A member of a set.

EQUATION -- An open sentence involving equality.

EXPONENT -- The particular use of a numeral to indicate how many times a certain number should be used as a factor. The expression 3^4 means $3 \times 3 \times 3 \times 3$ and the 4 is the exponent.

EXTRAPOLATION -- To calculate values outside an interval from values within the interval.

FACTOR -- One of the numerals in an indicated product is a factor of the product.

FORCE -- Force is a physical concept which can be described loosely as the push or pull on an object.

FULCRUM -- The point of support of a seesaw.

FUNCTION -- A function is a set of ordered pairs such that each element of the domain appears in one and only one ordered pair.

GRAPHICAL ANALYSIS -- To reach a conclusion by the use of graphs.

HYPOTHESIS -- In mathematics, an assumed proposition used as a premise in proving something else.

In science, a proposition held to be probably true because its consequences are found to be true.

INEQUALITY -- Any statement which indicates that one number or quantity is not equal to another is called an inequality.

INTEGERS -- The set of counting numbers, zero, and the additive inverses of the counting numbers make up the set of integers.

INTERCEPT -- The point on a number line at which a second line meets it.

INTERPOLATION -- To find an intermediate value between two given values.

LINEAR -- Pertaining to straight lines.

MASS -- Mass is a fundamental property of a body. It is <u>not</u> the same as the weight of the body. On the earth's surface, the weight of an object is proportional to its mass.

MATHEMATICAL MODEL -- A mathematical relation which represents the physical model. In most situations it will be an equation.

MAXIMUM VALUE -- The greatest value.

MEASUREMENT -- Any measurement is a process in which the object or event being measured is compared to the standard units for that object or event.

MULTIPLICATIVE INVERSE -- For every number, except zero, there is another number (called its multiplicative inverse) such that the product of the two numbers is one. For example, $\frac{1}{3}$ is the multiplicative inverse of 3 since $3 \times \frac{1}{3} = 1$.

NEGATIVE REAL NUMBERS -- The set of real numbers associated with points to the left of zero on the number line (where the unit point is to the right of zero) is the set of negative real numbers.

NON-INTEGRAL -- The property of not being an integer or not pertaining to an integer.

NUMBER PHRASE -- A number phrase is a name for a number.
An expression that represents a number.

NUMBER SENTENCE -- A statement about numbers and quantities.

NUMERICAL PHRASE -- A numerical phrase is any numeral given by an expression involving other numerals and signs of operation.

NUMERICAL SENTENCE -- A sentence which makes a statement about numbers.

OPEN PHRASE -- An open phrase is a phrase which does not represent a specific number

OPEN SENTENCE -- A mathematical sentence which contains one or more variables.

ORDERED PAIR -- A set containing exactly two elements, (a,b), in which one element is recognized as the first element.

PHYSICAL MODEL -- A single curve on a graph of the set of points which best represents a collection of data. It is an idealization of the behavior of a physical system.

POSITIVE REAL NUMBERS -- The set of real numbers greater than zero. Usually represented by the points to the right of zero on the number line.

POWER -- a^n is called a power of "a". More precisely, a^n is the nth power of "a".

QUADRANT -- One of the four regions into which the coordinate axes divide the coordinate plane.

RANGE -- The range is the set of second elements of the ordered pairs in a relation or function.

REAL NUMBERS -- The set of all numbers associated with points on the number line. A number which can be represented by a finite or infinite decimal expansion.

RECIPROCAL -- The multiplicative inverse of a real number is called the reciprocal of the number.

The reciprocal of a real number "a" ($a \neq 0$) is the number $\frac{1}{a}$. Zero has no reciprocal.

RELATION -- A relation is a set of ordered pairs. When the pair (x,y) is in the set and we use R to represent the relation, we say that x R y is true.

RESISTANCE -- The opposition to motion of a body by its surroundings.

SCALE -- In graphical representations the scale refers to the ratio in which the mapping represents the real situation.

SCIENTIFIC NOTATION -- The practice followed in mathematics and science of writing numbers as a number between one and ten multiplied by the appropriate power of ten. For example,

$$216 = 2.16 \times 10^{2}$$

$$0.0043 = 4.3 \times 10^{-3}$$

SLOPE -- The slope measures the steepness of the inclination of a line. It is the ratio of the rise to the run.

SOLUTION SET -- The set of elements in the domain of an open sentence which make the sentence true is called the solution set of the open sentence. Also called the truth set of the open sentence.

SUBSCRIPT -- A small letter or numeral written at the lower right of a symbol to distinguish it from other symbols of the same kind.

TERMINAL VELOCITY -- When the upward resistive force equals the downward gravitational pull on the object, terminal velocity has been reached.

TRANSLATION OF AXES -- Changing the coordinates of a set of points to coordinates referring to a new set of axes parallel to the original axes.

TRUTH SET -- The solution set of an open sentence is also called the truth set of that sentence.

See solution set.

VARIABLE -- A symbol which can be replaced by any member of a given set.

VELOCITY (CONSTANT) -- The slope of the line on a time-distance plot. It is given by $\frac{\text{distance}}{\text{time}}$.

VERB PHRASE -- The phrase that states the relationship involved between word phrases.

WORD PHRASE -- A mathematical phrase in word form.